Generalized Killing spinors on 3-Sasakian manifolds

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\((M, g)\) a (pseudo-) Riemannian spin manifold, \(\nabla^g\) the Levi-Civita and the induced spin connection

**Killing spinors**

\[
\nabla^g_X \Psi = a X \cdot \Psi
\] (1)

- **Killing number** \(a \in \mathbb{C} \sim \) eigenvalue of the Dirac operator
- 1\textsuperscript{st} integrability condition: \(\mathcal{R}_{X,Y} \Psi = -a^2 [X \cdot, Y \cdot] \Psi\)

\(\Rightarrow (M, g)\) is **Einstein** with \(\text{Scal} = 4 a^2 n(n - 1)\)

**Generalized Killing spinors**

\[
\nabla^g_X \Psi = S(X) \cdot \Psi
\] (2)

- \(S(X)\) is a section of \textit{symmetric endomorphisms} of \(TM\)
- **Not invariant!**
  (unless we consider \(S(X)\) as a part of the solution)
$V$ a vector bundle over $M$, $\nabla^V$ a linear connection in $V$, extend $\nabla^V$ by $\nabla^g$ also to $V$-valued differential forms

Let $\Phi \in \Omega^p(M, V)$ and $\Xi \in \Omega^{p+1}(M, V)$

**Killing(-Yano) forms**

$$\nabla^V_X \Phi = X \cdot \Xi \quad (3)$$

**$\ast$-Killing forms**

$$\nabla^V_X \Xi = X^b \wedge \Phi \quad (4)$$

**Special Killing forms**

$$\nabla^V_X \Phi = X \cdot \Xi, \quad \nabla^V_X \Xi = -c \, X^b \wedge \Phi \quad (5)$$

- Does **not** imply Einstein, but we have $\text{Scal} = c \, n(n-1)$
- *Cone construction*: spec. Killing $\Leftrightarrow$ parallel on the cone $\Rightarrow$ **Holonomy classification** via Berger’s list
Scalar-valued special Killing forms

All the examples where $M$ is compact are:

- **Round spheres**: a solution for arbitrary $(\Phi_0, \Xi_0)$
- **Sasakian**: $\Phi^{(k)} = \eta \wedge (d\eta)^k$, $\eta$ the contact form
- **Exceptional**: nearly Kähler in dim = 6, $G_2$ in dim = 7

Spinor-valued special Killing forms

$V = \Sigma$ is the spinor bundle and $\nabla^V$ the Killing spinor connection

\[ \nabla^V_X \Psi = \nabla^g_X \Psi - a X \cdot \Psi, \quad a \in \mathbb{C} \] (6)

- The cone construction works only when $c = 4a^2$.
- **Round spheres**: again a solution for arbitrary $(\Phi_0, \Xi_0)$
- Q: Can $c = 4a^2 = \frac{1}{n(n-1)} \text{Scal}$ be deduced in general from the integrability conditions?
2\textsuperscript{nd} order Killing spinors

≡ spinor-valued spec. K. 0-forms; combining the equations ⇔

\[
(\nabla^g)^2_{X,Y} \Psi = -a^2 X \cdot Y \cdot \Psi + \\
+ a (Y \cdot (\nabla^g_X \Psi) + X \cdot (\nabla^g_Y \Psi)) - c g(X, Y) \Psi
\]  \hspace{1cm} (7)

▶ 1\textsuperscript{st} integrability condition the same as for Killing spinors, so again ⇒ \((M, g)\) is \textbf{Einstein}.

▶ Includes Killing spinors with Killing number \(a' = -a\). ⇒ \textbf{Invariant} generalization of Killing spinors.

▶ Future research: Higher order equations derived from rank \(\geq 2\) symmetric Killing tensor-spinors.

Holonomy classification:

▶ 3-Sasakian: Admit an additional solution!

▶ Sasakian, G\(_2\): No additional solutions possible.

▶ Nearly Kähler: Remains to be checked.
Sasakian manifolds

\((M, g, \varphi, \xi, \eta)\), \(\dim M = 2m + 1\), such that:

- **almost contact:** \(\varphi^2 = -\text{Id}_{T M} + \eta \otimes \xi\), \(\eta(\xi) = 1\)
- **normal:** Nijenhuis torsion \(N_\varphi = [\varphi, \varphi] + d\eta \otimes \xi = 0\)
- **compatible metric:** \(g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y)\)
- **contact:** \(d\eta = 2\Phi\) where \(\Phi(X, Y) = g(X, \varphi(Y))\)

\(\Leftrightarrow |\xi| = 1, \eta = \xi^\flat\) is a **special Killing 1-form** with \(c = 1\)

\(\Rightarrow\) reduction of the structure group to \(U(m)\)

3-Sasakian manifolds

\((M, g, \varphi_i, \xi_i, \eta_i)\), \(\dim M = 4m + 3, i = 1, 2, 3\), such that each \((\varphi_i, \xi_i, \eta_i)\) is a Sasakian structure compatible with \(g\) and

\[
\begin{align*}
\varphi_k &= \varphi_i \varphi_j - \eta_j \otimes \xi_i = -\varphi_j \varphi_i + \eta_i \otimes \xi_j, \\
\xi_k &= \varphi_i \xi_j = -\varphi_j \xi_i, \\
\eta_k &= \eta_i \varphi_j = -\eta_j \varphi_i.
\end{align*}
\]

\(\Rightarrow\) reduction of the structure group to \(\text{Sp}(m)\); always **Einstein!**
3-(\(\alpha, \delta\))-Sasakian manifolds

Split \(TM = \mathcal{V} \oplus \mathcal{H}\), the vertical and horizontal distribution,

\[
\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle, \quad \mathcal{H} = \ker \eta_1 \cap \ker \eta_2 \cap \ker \eta_3.
\]

Rescale \(g\) on \(\mathcal{V}\) and \(\mathcal{H}\) \(\leadsto\) 2-parameter family of metrics \(\leadsto\) \(\leadsto\) 3-(\(\alpha, \delta\))-Sasakian manifolds

**Proposition**

\((M, g)\) is Einstein iff \(\delta = \alpha\) or \(\delta = (2m + 3) \alpha\).

**Dimension 7**

\[
\begin{array}{c|c|c}
\delta = \alpha = 1 & g = g_1 & \text{the original 3-Sasakian structure} \\
\delta = 5\alpha & \tilde{g} = g_{1:5} & \text{canonical cocalibrated } G_2\text{-structure}
\end{array}
\]

- The cocalibrated \(G_2\)-structure with metric \(\tilde{g}\) possesses the so called **canonical spinor** satisfying \(\nabla^c \Psi_0 = 0\).
- With respect to the original 3-Sasakian metric \(g\) the spinor field \(\Psi_0\) becomes a **generalized Killing spinor**.
Canonical spinor

- **Ψ₀** is a **generalized Killing spinor**.

\[
\nabla^g_\xi \Psi_0 = \frac{1}{2} \xi \cdot \Psi_0, \quad \nabla^g_Y \Psi_0 = -\frac{3}{2} Y \cdot \Psi_0, \quad \xi \in \mathcal{V}, \ Y \in \mathcal{H}
\]  
(8)

- Reeb vector fields \( \xi_i \) are **special Killing** with \( c = 1 \).

- **Ψᵢ = \( \xi_i \cdot \Psi_0 \)** are **Killing spinors** with \( a = \frac{1}{2} \).

\[
\nabla^g_X \Psi_i = \frac{1}{2} X \cdot \Psi_i, \quad X \in TM; \ i = 1, 2, 3
\]  
(9)

**Proposition**

**Ψ₀** is also a **2nd order Killing spinor** with \( a = -\frac{1}{2} \) and \( c = 1 \) which is not a Killing spinor.

- Invariant description of the canonical spinor **Ψ₀**.

- **WIP:** Describe **Ψ₀** in general for \( \dim M = 4m + 3 \) without the detour to the G₂-structure.
References

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THANK YOU FOR YOUR ATTENTION!