Nichols algebras from quantum principal bundles over quantum flag manifolds

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Notation

Let H be a Hopf algebra

- $\varepsilon\colon H \to \mathbb{C}$
- $\Delta\colon H \to H \otimes H$, $\Delta h = \sum_i x_i \otimes y_i = h_{(1)} \otimes h_{(2)}$
- $S\colon H \to H$

Let $H^+ = H \cap \ker \varepsilon$ and $h^+ = h - \varepsilon(h)1$ for $h \in H$.

Nichols algebras

A *braided vector* space is a pair (V, σ) where V is a vector space and $\sigma \in \mathrm{Aut}(V \otimes V)$ such that

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}, \qquad \sigma_{12} := \sigma \otimes \mathrm{id}, \ \sigma_{23} := \mathrm{id} \otimes \sigma.$$

Let $\mathbb{B}_n$ be the braid group of n stands generated by $\beta_1, \dots, \beta_n$ subject to the relations

$$\begin{aligned} \beta_i\beta_{i+1}\beta_i &= \beta_{i+1}\beta_i\beta_{i+1}, & 1 \le i, j \le n-2; \\ \beta_i\beta_j &= \beta_j\beta_i, & 1 \le i, j \le n-2, |i-j| \ge 2. \end{aligned}$$

$$\rho_n : \mathbb{B}_n \to \mathrm{Aut}(V^{\otimes n}), \quad \rho_n(\beta_i) = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \sigma \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id},$$

$$s \colon S_n \to \mathbb{B}_n \quad \text{such that} \quad s(t_i) = \beta_i, \ s(t_i t_{i+1}) = s(t_i)s(t_{i+1})$$

$$\mathfrak{S}_n^\sigma := \sum_{\pi \in S_n} \rho_n(s(\pi)) \colon V^{\otimes n} \to V^{\otimes n}.$$

Definition

The Nichols algebra of a braided vector space (V, σ) is the braided Hopf algebra defined by

$$\mathfrak{B}(V) := \bigoplus_{n \in \mathbb{Z}_{n\geq 0}} \mathfrak{B}_n(V), \quad \text{where} \quad \mathfrak{B}_n(V) = \mathcal{T}^n(V)/\ker(\mathfrak{S}_n^\sigma).$$

Example

$\sigma(v \otimes w) = w \otimes v$ for all $v, w \in V$ then $\mathfrak{B}(V) = \Lambda(V)$.

Example

$\sigma(v \otimes w) = -w \otimes v$ for all $v, w \in V$ then $\mathfrak{B}(V) = S(V)$.

Example

For $U_q(\mathfrak{g})$, $U_q(\mathfrak{n}_+)$ and $U_q(\mathfrak{n}_-)$ are Nichols algebras.

Definition

A *Yetter–Drinfeld module* over a Hopf algebra H is an H-module V, with an action $\triangleleft$, and a H-comodule structure such that

$$v_{(0)}\triangleleft h_{(1)}\otimes v_{(1)}\triangleleft h_{(2)} = (v\triangleleft h_{(2)})_{(0)}\otimes h_{(1)}(v\triangleleft h_{(2)})_{(1)} \quad \forall h \in H,\ v \in V.$$

The category of Yetter–Drinfeld modules over H is denoted by YD_H^H.

The braiding in the category YD_H^H is defined by

$$\sigma\colon V\otimes W \to W\otimes V, \qquad v\otimes w \mapsto w_{(0)}\otimes v\triangleleft w_{(1)} \quad \text{for } v \in V,\ w \in W.$$

Example

$H = \mathbb{C}\{1\}$, $v \triangleleft 1 = v$ for all $v \in V$ then $\mathfrak{B}(V) = S(V)$.

Example

$H = \mathbb{C}\{\mathbb{Z}/2\} = \mathrm{Span}\{1, -1\}$, $v \triangleleft (-1) = -v$ then $\mathfrak{B}(V) = \Lambda(V)$.

Let G and H be Hopf algebras, and $\pi : G \to H$ a surjective Hopf algebra map. A right H-coaction, giving G the structure of a right H-comodule algebra, is given by

$$\Delta_R := (\mathrm{id} \otimes \pi) \circ \Delta : G \to G \otimes H.$$

We call the coinvariant subspace $M := G^{\mathrm{co}\, H}$ of such a coaction a *quantum homogeneous space*.
A *strong bicovariant splitting map* is a unital linear map $i : H \to G$ splitting the projection $\pi : G \to H$ such that

$$(i \otimes \mathrm{id}) \circ \Delta = \Delta_R \circ i, \qquad (\mathrm{id} \otimes i) \circ \Delta = \Delta_L \circ i.$$

Remark

The coproduct of G restricts to a left M-comodule $\Delta_L : M \to G \otimes M$, giving M the structure of a left G-comodule map.

Let ${}^H\mathrm{Mod}$ denote the category of left H-comodules.

Definition

Let ${}^G_M\mathrm{Mod}_0$ be the category whose objects are left G-comodules $\Delta_L : \mathcal{F} \to G \otimes \mathcal{F}$, endowed with a M-M-bimodule structure, such that

1. $\Delta_L(mf) = \Delta_L(m)\Delta_L(f)$, for all $f \in \mathcal{F}, m \in M$,
2. $\mathcal{F}B^+ = B^+\mathcal{F}$,

and whose morphisms are left G-comodule, M-M-bimodule, maps.

Denoting by $\Box_H$ the cotensor product over H.

$$\Phi : {}^G_M\mathrm{Mod}_0 \to {}^H\mathrm{Mod}, \qquad \mathcal{F} \mapsto \mathcal{F}/M^+\mathcal{F},$$
$$\Psi : {}^H\mathrm{Mod} \to {}^G_M\mathrm{Mod}, \qquad V \mapsto G\,\Box_H V,$$

where the left H-comodule structure of $\Phi(\mathcal{F})$ is given by $(\pi \otimes \mathrm{id}) \circ \Delta_L$, and the M-M-module, and left G-comodule, structures of $\Psi(V)$ are defined on the first tensor factor.

Definition

A differential calculi is a dg-algebra $(\Omega^\bullet \simeq \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \Omega^k, \mathrm{d})$ which is generated in degree 0 as a dg-algebra.

Definition

A first-order differential calculi over an algebra A is a pair (Ω^1, d), where Ω^1 is A-A-bimodule and $\mathrm{d}\colon A \to \Omega^1$ is a linear map such that

1) $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ for all $a, b \in A$,

2) Ω^1 is generated as a left A-module by

$$\mathrm{d}A = \mathrm{Span}(\mathrm{d}a \mid a \in A).$$

Universal Differential Calculi

Definition

The universal first-order calculi over A is the pair $(\Omega^1_u(A), \mathrm{d}_u)$, where

1. $\Omega^1_u(A) = \ker m$, where $m\colon A \otimes A \to A$ it the product map,
2. $\mathrm{d}_u\colon a \mapsto 1 \otimes a - a \otimes 1$ for $a \in A$.

Theorem (Woronowicz'89)

Every first-order differential calculi over A is of the form $(\Omega^1_u(A)/N, \mathrm{proj} \circ \mathrm{d}_u)$, where N is a A-sub-bimodule of $\Omega^1_u(A)$, and $\mathrm{proj}\colon \Omega^1_u(A) \to \Omega^1_u(A)/N$ is the canonical projection.

For $\Omega^1 = \Omega^1_u(A)/N$ the maximal prolongation is the differential calculi over A given by the pair

$$(\Omega^\bullet(A) := \mathcal{T}(A)/\langle N + \mathrm{d}N \rangle, \mathrm{d}).$$

Covariant Differential Calculi

A first-order differential calculi $(\Omega^1(M), \mathrm{d})$ over a quantum homogeneous space $M = G^{\mathrm{co}\,H}$ is *covariant* if there exists a map $\Delta_L \colon \Omega^1(M) \to G \otimes \Omega^1(M)$ such that

$$\Delta_L(m\mathrm{d}n) = \Delta(m)(\mathrm{id} \otimes \mathrm{d})\Delta(n) \quad \text{for all } m, n \in M.$$

Note that $\Omega^1(M)$ is an object in ${}^G_M\mathrm{Mod}_M$. Moreover, $(\Omega_u(M), \mathrm{d}_u)$ is covariant and any covariant calculi $\Omega^1(M) \simeq \Omega^1(M)/N$ is classified by the sub-object N of $\Omega^1_u(M)$ in ${}^G_M\mathrm{Mod}_M$.
Warning. d is not a morphism in ${}^G_M\mathrm{Mod}_M$.

Denote by ${}^H\mathcal{I}(M^+)$ the category whose objects I are sub-comodules of M, satisfying

$$bm^+ \in I \quad \text{for all } m \in M,\, b \in I$$

and for two objects I, J, the hom-set $\mathrm{Hom}(I,J)$ is comprised of the inclusion map if $I \subseteq J$, and is otherwise empty.

Theorem

An equivalence of categories between ${}^H\mathcal{I}(M^+)$ and ${}^H\mathsf{FODC}(M)$ is given by functors

$$I\colon {}^H\mathsf{FODC}(M) \to {}^H\mathcal{I}(M^+), \quad \Omega^1_u(M)/N \mapsto \{\varepsilon(a_i)b_i^+ \mid \sum_i a_i \mathrm{d}b_i \in N\}$$

$$K\colon {}^H\mathcal{I}(M^+) \to {}^H\mathsf{FODC}(M), \qquad I \mapsto (\Phi(M^+/I), \mathrm{d}), \quad I \in {}^H\mathcal{I}(M^+)$$

where the exterior derivative is defined according to

$$\mathrm{d} : M \to \Phi(M^+/I), \quad m \mapsto m_{(1)} \otimes [m^+_{(2)}] \qquad \textit{for } m \in M,$$

Principal Comodule Algebras

We say that a left H-comodule algebra P is a H-Hopf–Galois extension of $M := P^{\mathrm{co}(H)}$ if the following map is an isomorphism

$$\mathrm{can} := (m_P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R) : P \otimes_M P \to P \otimes H.$$

where m_P is the multiplication in P.

Definition

A *principal right H-comodule algebra* is a right H-comodule algebra (P, Δ_R) such that P is a H-Hopf–Galois extension of M and P is faithfully flat as a right and left M-module.

For a right H-comodule algebra (P, Δ_R) with $M := P^{\mathrm{co}\,H}$ we have that the extension $M \to P$ being Hopf–Galois extension is equivalent to exactness of the sequence

$$0 \longrightarrow P\Omega^1_u(M)P \xrightarrow{\iota} \Omega^1_u(P) \xrightarrow{\mathrm{ver}} P \otimes H^+ \longrightarrow 0,$$

where $\Omega^1_u(M)$ is the restriction of $\Omega^1_u(P)$ to M, ι is the inclusion map.

Definition

A *quantum principal bundle* is a triple (P, H, N) consisting of a H-Hopf–Galois extension $\Delta_R : P \to P \otimes H$ of $M = P^{coH}$ together with a sub-P-P-bimodule $N \subseteq \Omega^1_u(P)$ which is coinvariant under the right H-coaction Δ_R and for which there exists an Ad_H-coinvariant right ideal $I \subseteq H^+$ satisfying $\mathrm{ver}(N) = M \otimes I$.

Quantum Principal Bundles

$$\begin{array}{ccc} \Omega^1_u(G) & \xrightarrow{\quad\overline{\mathrm{ver}}\quad} & \\ \Big\downarrow{\scriptstyle \mathrm{unit}_G} & \searrow & \\ G \otimes G^+ & \xrightarrow[\mathrm{id}\,\otimes\pi]{} & G \,\square_H H^+. \end{array}$$

where $\overline{\mathrm{ver}}$ is the restriction of ver to $\Omega^1_u(G)$. This means that for any sub-$G - G$-bimodule $N \subseteq \Omega^1_u(G)$, with corresponding ideal $\mathrm{unit}_G(N) = I$, it holds that

$$\mathrm{ver}(N) = G \otimes \pi(I).$$

From this we see that the requirement that $\mathrm{ver}(N) = M \otimes I$ is automatically satisfied. Meaning that a quantum principal homogeneous space can equivalently described as a homogeneous Hopf–Galois extension, together with an Ad_R-coideal of G^+.

Theorem

Let $\pi : G \to H$ define a quantum principal bundle with $M := G^{\mathrm{co}\,H}$ and $V_M := \Phi(\Omega^1(M))$. A right H-action is defined by

$$\triangleleft : V_M \otimes H \to V_M, \qquad \big([m], h\big) \mapsto [m \cdot i(h)], \quad m \in M, h \in H,$$

a right H-coaction is defined by

$$\mathrm{Ad}_R\colon M \to M{\otimes}H, \qquad \mathrm{Ad}_R\colon m \to m_{(2)}{\otimes}\pi(S(m_{(1)})m_{(3)}), \quad m \in M$$

The module structure is independent of the choice of i, and a Yetter–Drinfeld module is given by the triple $(V_M, \triangleleft, \mathrm{Ad}_R)$.

Quantised Coordinate Algebras $\mathcal{O}_q(G)$

Let V be a finite-dimensional $U_q(\mathfrak{g})$-module, $v \in V$, and $f \in V^*$, the linear dual of V. Consider the function

$$c^V_{v,f} : U_q(\mathfrak{g}) \to \mathbb{C}, \qquad X \mapsto f\big(X(v)\big).$$

The *coordinate ring* of V is the subspace

$$C(V) := \mathrm{Span}_{\mathbb{C}}\{c^V_{v,f} \,|\, v \in V,\ f \in V^*\} \subseteq U_q(\mathfrak{g})^*.$$

In fact, we see that $C(V) \subseteq U_q(\mathfrak{g})^\circ$, where $U_q(\mathfrak{g})^\circ$ denotes the Hopf dual of a Hopf algebra $U_q(\mathfrak{g})$, and that a Hopf subalgebra of $U_q(\mathfrak{g})^\circ$ is given by

$$\mathcal{O}_q(G) := \bigoplus_{V \in \mathsf{Rep}_1 U_q(\mathfrak{g})} C(V).$$

We call $\mathcal{O}_q(G)$ the *quantum coordinate algebra of* G, where G is the unique connected, simply connected, complex algebraic group having $\mathfrak{g}$ as its complex Lie algebra.

Quantum Flag Manifolds

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank r. For S a subset of simple roots, consider the Hopf subalgebra

$$U_q(\mathfrak{l}_S) := \big\langle K_i, E_j, F_j \,|\, i = 1, \dots, r; \alpha_j \in S \big\rangle.$$

From the Hopf algebra embedding $\iota : U_q(\mathfrak{l}_S) \hookrightarrow U_q(\mathfrak{g})$, we get the dual Hopf algebra map $\iota^\circ : U_q(\mathfrak{g})^\circ \to U_q(\mathfrak{l}_S)^\circ$. We have

$$\pi_S := \iota^\circ|_{\mathcal{O}_q(G)} : \mathcal{O}_q(G) \to U_q(\mathfrak{l}_S)^\circ,$$

and the Hopf subalgebra $\mathcal{O}_q(L_S) := \pi_S\big(\mathcal{O}_q(G)\big) \subseteq U_q(\mathfrak{l}_S)^\circ$. The quantum-homogeneous space

$$\pi : \mathcal{O}_q(G) \to \mathcal{O}_q(L_S), \qquad (1)$$

is called the *quantum flag manifold associated* to S and denoted by

$$\mathcal{O}_q\big(G/L_S\big) := \mathcal{O}_q\big(G\big)^{\mathrm{co}(\mathcal{O}_q(L_S))}.$$

The extension $\mathcal{O}_q(G/L_S) \hookrightarrow \mathcal{O}_q(G)$ is a principal comodule algebra.

Theorem (Heckenberger–Kolb)

For S a subset of simple roots corresponding to the classical irreducible flag manifolds, there exist exactly two non-isomorphic, irreducible, left-covariant, finite-dimensional, first-order differential calculi of finite dimension over $\mathcal{O}_q(G/L_S)$. Moreover, the corresponding maximal prolongations have classical dimensions.

We denote the direct sum of the corresponding calculi by

$$\Omega^1_{HK}(G/L_S) = \Omega^{(1,0)}_{HK}(G/L_S) \oplus \Omega^{(0,1)}_{HK}(G/L_S).$$

In this case we have $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{l}_S = \mathfrak{sl}_r \oplus \mathfrak{gl}_{n-r}$.
Denote the corresponding quantized coordinate algebra as $\mathcal{O}_q(\mathsf{Gr}_{n,r}) = \mathcal{O}_q(SU_n)^{\mathrm{co}\,\mathcal{O}_q(SU_r \times U_{n-1})}$.

Fact. The quantum principal bundle given by $(\mathcal{O}_q(\mathsf{Gr}_{n,r}) \hookrightarrow \mathcal{O}_q(SU_n), \Omega^1_{bc}(SU_n))$ does not gives $\Omega^1_{HK}(\mathsf{Gr}_{n,r})$.

Proposition

There is an ideal N such that the restriction of $\Omega^1_u(SU_n)/N$ to $\mathcal{O}_q(\mathsf{Gr}_{n,r})$ is $\Omega^1_{HK}(\mathsf{Gr}_{n,r})$ and $(\mathcal{O}_q(\mathsf{Gr}_{n,r}) \hookrightarrow \mathcal{O}_q(SU_n), \Omega^1_{bc}(SU_n)/N)$ defines a quantum principal bundle.

Main Theorem: Quantum Grassmannian Case

Theorem

The maximal prolongations of $\Omega^{(0,1)}_{HK}(\mathsf{Gr}_{n,r})$ *and* $\Omega^{(1,0)}_{HK}(\mathsf{Gr}_{n,r})$ *are Nichols algebras. The corresponding Yetter–Drinfeld module structures are given by the quantum principal bundle* $(\mathcal{O}_q(\mathsf{Gr}_{n,r}) \hookrightarrow \mathcal{O}_q(SU_n), \Omega^1_{bc}(SU_n)/N)$.

Sketch of Proof

- Let $V = \Phi(\Omega_{HK}^{(1,0)}(\mathrm{Gr}_{n,r}))$ then $\Phi(\Omega_{HK}^{(0,1)}(\mathrm{Gr}_{n,r})) = V^*$ as $U_q(\mathfrak{sl}_r \oplus \mathfrak{gl}_{n-r})$-module. Moreover, $V = W_1 \otimes W_2$ where W_1 is a tautological $U_q(\mathfrak{sl}_r)$-module and W_2 is a tautological $U_q(\mathfrak{gl}_{n-r})$-module.

$$V \otimes V = \text{“}\Lambda^2 W_1\text{”} \otimes \text{“}\Lambda^2 W_2\text{”} \bigoplus \text{“}S^2 W_1\text{”} \otimes \text{“}S^2 W_2\text{”}.$$
$$\bigoplus \text{“}S^2 W_1\text{”} \otimes \text{“}\Lambda^2 W_2\text{”} \bigoplus \text{“}\Lambda^2 W_1\text{”} \otimes \text{“}S^2 W_2\text{”}.$$

- For the maximal prolongation $\Omega_{HK}^{(\bullet,0)}(\mathrm{Gr}_{n,r})$ of $\Omega_{HK}^{(1,0)}(\mathrm{Gr}_{n,r})$

$$\Phi(\Omega_{HK}^{(\bullet,0)}(\mathrm{Gr}_{n,r})) = \Lambda_q V := \mathcal{T}(V)/\langle \Lambda_q^2 V \rangle.$$

- Quantum Howe duality

$$\Lambda_q(V) = \Lambda_q(W_1 \otimes W_2) \simeq \bigoplus_{\lambda} M_\lambda \otimes M_{\lambda^t}, \tag{2}$$

where λ varies over all r-bounded (partitions) weights of $U_q(\mathfrak{sl}_r)$, λ^t is the transpose of λ.