Quantum Flags, Quantum Line Bundles
and the Quantum Duality Principle

Rita Fioresi, Prague, September 19, 2019
Hypothesis:

- $G$ semisimple complex algebraic group.
Hypothesis:

- $G$ semisimple complex algebraic group.
- $P$ a closed algebraic subgroup of $G$ ($P$ parabolic).
Homogeneous Projective Varieties

Hypothesis:

- $G$ semisimple complex algebraic group.
- $P$ a closed algebraic subgroup of $G$ ($P$ parabolic).
- $\chi : P \rightarrow \mathbb{C}^\times$ a character of $P$. 

Rita Fioresi
Quantum flags, quantum line bundles and the Quantum Duality Principle
September 18, 2019
Homogeneous Projective Varieties

Hypothesis:

- $G$ semisimple complex algebraic group.
- $P$ a closed algebraic subgroup of $G$ ($P$ parabolic).
- $\chi : P \to \mathbb{C}^\times$ a character of $P$.
- $\mathcal{L}$ is a line bundle on $G/P$ associated with $\chi$, with global sections:
Homogeneous Projective Varieties

Hypothesis:

- $G$ semisimple complex algebraic group.
- $P$ a closed algebraic subgroup of $G$ ($P$ parabolic).
- $\chi : P \to \mathbb{C}^\times$ a character of $P$.
- $\mathcal{L}$ is a line bundle on $G/P$ associated with $\chi$, with global sections:

$$\mathcal{O}(G/P)_1 = \{ f : G \to \mathbb{C} \mid f(gh) = \chi^{-1}(h)f(g) \}$$
Homogeneous Projective Varieties

Hypothesis:

- $G$ semisimple complex algebraic group.
- $P$ a closed algebraic subgroup of $G$ ($P$ parabolic).
- $\chi : P \longrightarrow \mathbb{C}^\times$ a character of $P$.
- $L$ is a line bundle on $G/P$ associated with $\chi$, with global sections:
  \[ \mathcal{O}(G/P)_1 = \{ f : G \longrightarrow \mathbb{C} \mid f(gh) = \chi^{-1}(h)f(g) \} \]

- very ample $L$ gives a projective embedding of $G/P$.  

Rita Fioresi
Quantum flags, quantum line bundles and the Quantum Duality Principle
September 18, 2019 2 / 21
Hypothesis:

- $G$ semisimple complex algebraic group.
- $P$ a closed algebraic subgroup of $G$ ($P$ parabolic).
- $\chi: P \to \mathbb{C}^\times$ a character of $P$.
- $\mathcal{L}$ is a line bundle on $G/P$ associated with $\chi$, with global sections:

$$\mathcal{O}(G/P)_1 = \{ f : G \to \mathbb{C} \mid f(gh) = \chi^{-1}(h)f(g) \}$$

- very ample $\mathcal{L}$ gives a projective embedding of $G/P$.

The coordinate (graded) algebra of the supervariety $G/P$ with respect to the projective embedding is
Homogeneous Projective Varieties

Hypothesis:

- $G$ semisimple complex algebraic group.
- $P$ a closed algebraic subgroup of $G$ ($P$ parabolic).
- $\chi : P \rightarrow \mathbb{C}^\times$ a character of $P$.
- $\mathcal{L}$ is a line bundle on $G/P$ associated with $\chi$, with global sections:

\[ \mathcal{O}(G/P)_1 = \{ f : G \rightarrow \mathbb{C} \mid f(gh) = \chi^{-1}(h)f(g) \} \]

- very ample $\mathcal{L}$ gives a projective embedding of $G/P$.

The coordinate (graded) algebra of the supervariety $G/P$ with respect to the projective embedding is

\[ \mathcal{O}(G/P) := \sum \mathcal{O}(G/P)_n, \quad \text{where} \]

\[ \mathcal{O}(G/P)_n := \{ f : G \rightarrow \mathbb{C} \mid f(gh) = \chi^{-n}(h)f(g) \}. \]
Example: The Projective space $\mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$
Example: The Projective space $\mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$

Here

$$G = \text{SL}_2(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad \mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$$
Example: The Projective space $\mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$

Here

$$G = \text{SL}_2(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad \mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$$

We define the character of $P$: 
Example: The Projective space $\mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$

Here

$$G = \text{SL}_2(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad \mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$$

We define the character of $P$:

$$\chi : P \longrightarrow \mathbb{C}^\times, \quad \chi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = a^{-1}$$

The corresponding line bundle has sections:
Example: The Projective space $\mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$

Here

$$G = \text{SL}_2(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad \mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$$

We define the character of $P$:

$$\chi : P \longrightarrow \mathbb{C}^\times, \quad \chi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = a^{-1}$$

The corresponding line bundle has sections:

$$\mathcal{O}(\mathbb{P}^1)_1 = \{ \hat{a}, \hat{c} : \text{SL}_2(\mathbb{C}) \longrightarrow \mathbb{C} \}$$

$$\mathcal{O}(\mathbb{P}^1) = \mathbb{C}[\hat{a}, \hat{c}] \subset \mathcal{O}(\text{SL}_2) = \mathbb{C}[\hat{a}, \hat{b}, \hat{c}, \hat{d}]/(\hat{a}\hat{d} - \hat{c}\hat{b} - 1)$$

In fact
Example: The Projective space $\mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$

Here

$G = \text{SL}_2(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad \mathbb{P}^1(\mathbb{C}) = \text{SL}_2(\mathbb{C})/P$

We define the character of $P$:

$\chi : P \rightarrow \mathbb{C}^\times, \quad \chi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = a^{-1}$

The corresponding line bundle has sections:

$\mathcal{O}(\mathbb{P}^1)_1 = \{ \hat{a}, \hat{c} : \text{SL}_2(\mathbb{C}) \rightarrow \mathbb{C} \}$

$\mathcal{O}(\mathbb{P}^1) = \mathbb{C}[\hat{a}, \hat{c}] \subset \mathcal{O}(\text{SL}_2) = \mathbb{C}[\hat{a}, \hat{b}, \hat{c}, \hat{d}]/(\hat{a}\hat{d} - \hat{c}\hat{b} - 1)$

In fact

$\hat{a} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = aa' = \chi \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}^{-1} \hat{a} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. 

Rita Fioresi
Quantum flags, quantum line bundles and the Quantum Duality Principle
September 18, 2019 3 / 21
Example: The Grassmannian $G(2, 4)$
Example: The Grassmannian $G(2, 4)$

Here

$$G = \text{SL}_4(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad G(2, 4) = \text{SL}_4(\mathbb{C})/P$$
Example: The Grassmannian $G(2, 4)$

Here

$$G = \text{SL}_4(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad G(2, 4) = \text{SL}_4(\mathbb{C})/P$$

We define the character of $P$: 
Example: The Grassmannian $G(2, 4)$

Here

$$G = \text{SL}_4(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad G(2, 4) = \text{SL}_4(\mathbb{C})/P$$

We define the character of $P$:

$$\chi : P \longrightarrow \mathbb{C}^\times, \quad \chi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \text{det}(a)^{-1}$$

The corresponding line bundle has sections:
Example: The Grassmannian $G(2, 4)$

Here

$$G = SL_4(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad G(2, 4) = SL_4(\mathbb{C})/P$$

We define the character of $P$:

$$\chi : P \rightarrow \mathbb{C}^\times, \quad \chi \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \det(a)^{-1}$$

The corresponding line bundle has sections:

$$O(G(2, 4))_1 = \{ d_{ij} : SL_4(\mathbb{C}) \rightarrow \mathbb{C} | d_{ij} = a_{i1}a_{j2} - a_{j1}a_{i2} \},$$

$$O(G/P) = \mathbb{C}[d_{ij}] \subset \mathbb{C}[G]$$
Example: The Grassmannian $G(2, 4)$

Here

$$G = \text{SL}_4(\mathbb{C}), \quad P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\}, \quad G(2, 4) = \text{SL}_4(\mathbb{C})/P$$

We define the character of $P$:

$$\chi : P \rightarrow \mathbb{C}^\times, \quad \chi \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = \text{det}(a)^{-1}$$

The corresponding line bundle has sections:

$$\mathcal{O}(G(2, 4))_1 = \{ d_{ij} : \text{SL}_4(\mathbb{C}) \rightarrow \mathbb{C} | d_{ij} = a_{i1}a_{j2} - a_{j1}a_{i2} \},$$

$$\mathcal{O}(G/P) = \mathbb{C}[d_{ij}] \subset \mathbb{C}[G]$$

$$\hat{d}_{ij} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \left( \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \right) = d_{ij} \text{det}(a') = \chi \left( \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \right)^{-1} \hat{a} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$
Key observations

- Since $\chi \in \mathbb{C}[P]$, the property $f(gh) = \chi(h)^{-1}f(g)$ in terms of Hopf algebras reads:
Line Bundles and projective embeddings

Key observations

- Since $\chi \in \mathbb{C}[P]$, the property $f(gh) = \chi(h)^{-1}f(g)$ in terms of Hopf algebras reads:

  $$(1 \otimes \pi)\Delta(f) = \chi^{-1} \otimes f, \quad \text{for} \quad \pi : \mathbb{C}[G] \rightarrow \mathbb{C}[P]$$

- The element $\chi^{-1} \in \mathbb{C}[P]$ allows us to retrieve the line bundle.

**Proposition.** *If $G/H$ is embedded in $\mathbb{P}^m$ via a line bundle, then there exists $t \in \mathbb{C}[G]$ such that $\pi(t) = \chi^{-1}$ and with the property:*
Line Bundles and projective embeddings

Key observations

- Since $\chi \in \mathbb{C}[P]$, the property $f(gh) = \chi(h)^{-1}f(g)$ in terms of Hopf algebras reads:

  $$(1 \otimes \pi)\Delta(f) = \chi^{-1} \otimes f, \quad \text{for} \quad \pi : \mathbb{C}[G] \rightarrow \mathbb{C}[P]$$

- The element $\chi^{-1} \in \mathbb{C}[P]$ allows us to retrieve the line bundle.

**Proposition.** *If $G/H$ is embedded in $\mathbb{P}^m$ via a line bundle, then there exists $t \in \mathbb{C}[G]$ such that $\pi(t) = \chi^{-1}$ and with the property:

  $$(1 \otimes \pi)\Delta(t) = t \otimes \pi(t)$$

This $t$ is defined up to a constant and determines uniquely the line bundle, hence the projective embedding of $G/H$:***
Key observations

- Since $\chi \in \mathbb{C}[P]$, the property $f(gh) = \chi(h)^{-1}f(g)$ in terms of Hopf algebras reads:

  $$(1 \otimes \pi)\Delta(f) = \chi^{-1} \otimes f,$$

  for $\pi : \mathbb{C}[G] \rightarrow \mathbb{C}[P]$

- The element $\chi^{-1} \in \mathbb{C}[P]$ allows us to retrieve the line bundle.

**Proposition.** *If $G/H$ is embedded in $\mathbb{P}^m$ via a line bundle, then there exists $t \in \mathbb{C}[G]$ such that $\pi(t) = \chi^{-1}$ and with the property:

  $$(1 \otimes \pi)\Delta(t) = t \otimes \pi(t)$$

This $t$ is defined up to a constant and determines uniquely the line bundle, hence the projective embedding of $G/H$:

$$\mathcal{O}(G/P)_n = \{f \in \mathcal{O}(G) \mid (1 \otimes \pi)\Delta(f) = f \otimes \pi(t^n)\}$$
Key observations

- Since $\chi \in \mathbb{C} [P]$, the property $f(gh) = \chi(h)^{-1}f(g)$ in terms of Hopf algebras reads:

  $$ (1 \otimes \pi) \Delta(f) = \chi^{-1} \otimes f, \quad \text{for} \quad \pi : \mathbb{C} [G] \rightarrow \mathbb{C} [P] $$

- The element $\chi^{-1} \in \mathbb{C} [P]$ allows us to retrieve the line bundle.

Proposition. If $G/H$ is embedded in $\mathbb{P}^m$ via a line bundle, then there exists $t \in \mathbb{C} [G]$ such that $\pi(t) = \chi^{-1}$ and with the property:

  $$ (1 \otimes \pi) \Delta(t) = t \otimes \pi(t) $$

This $t$ is defined up to a constant and determines uniquely the line bundle, hence the projective embedding of $G/H$:

$$ \mathcal{O}(G/P)_n = \{ f \in \mathcal{O}(G) \mid (1 \otimes \pi) \Delta(f) = f \otimes \pi(t^n) \} $$

Example.
Key observations

Since \( \chi \in \mathbb{C}[P] \), the property \( f(gh) = \chi(h)^{-1}f(g) \) in terms of Hopf algebras reads:

\[
(1 \otimes \pi)\Delta(f) = \chi^{-1} \otimes f,
\]

for \( \pi : \mathbb{C}[G] \rightarrow \mathbb{C}[P] \).

The element \( \chi^{-1} \in \mathbb{C}[P] \) allows us to retrieve the line bundle.

**Proposition.** If \( G/H \) is embedded in \( \mathbb{P}^m \) via a line bundle, then there exists \( t \in \mathbb{C}[G] \) such that \( \pi(t) = \chi^{-1} \) and with the property:

\[
(1 \otimes \pi)\Delta(t) = t \otimes \pi(t)
\]

This \( t \) is defined up to a constant and determines uniquely the line bundle, hence the projective embedding of \( G/H \):

\[
\mathcal{O}(G/P)_n = \{ f \in \mathcal{O}(G) | (1 \otimes \pi)\Delta(f) = f \otimes \pi(t^n) \}
\]

**Example.** For \( G/P = G(2, 4) \) and the Plucker embedding,

\[
t = a_{11}a_{22} - a_{21}a_{12},
\]
**Key observations**

- Since $\chi \in \mathbb{C}[P]$, the property $f(gh) = \chi(h)^{-1}f(g)$ in terms of Hopf algebras reads:

  $$(1 \otimes \pi)\Delta(f) = \chi^{-1} \otimes f, \quad \text{for} \quad \pi : \mathbb{C}[G] \rightarrow \mathbb{C}[P]$$

- The element $\chi^{-1} \in \mathbb{C}[P]$ allows us to retrieve the line bundle.

**Proposition.** If $G/H$ is embedded in $\mathbb{P}^m$ via a line bundle, then there exists $t \in \mathbb{C}[G]$ such that $\pi(t) = \chi^{-1}$ and with the property:

  $$(1 \otimes \pi)\Delta(t) = t \otimes \pi(t)$$

This $t$ is defined up to a constant and determines uniquely the line bundle, hence the projective embedding of $G/H$:

$$\mathcal{O}(G/P)_n = \{f \in \mathcal{O}(G) \mid (1 \otimes \pi)\Delta(f) = f \otimes \pi(t^n)\}$$

**Example.** For $G/P = G(2, 4)$ and the Plucker embedding, $t = a_{11}a_{22} - a_{21}a_{12}$, $\mathcal{O}(G/P) = \mathbb{C}[d_{ij}] \subset \mathbb{C}[G]$
Line Bundles and projective embeddings

Key observations

Since \( \chi \in \mathbb{C}[P] \), the property \( f(gh) = \chi(h)^{-1}f(g) \) in terms of Hopf algebras reads:

\[
(1 \otimes \pi)\Delta(f) = \chi^{-1} \otimes f, \quad \text{for} \quad \pi : \mathbb{C}[G] \rightarrow \mathbb{C}[P]
\]

The element \( \chi^{-1} \in \mathbb{C}[P] \) allows us to retrieve the line bundle.

**Proposition.** If \( G/H \) is embedded in \( \mathbb{P}^m \) via a line bundle, then there exists \( t \in \mathbb{C}[G] \) such that \( \pi(t) = \chi^{-1} \) and with the property:

\[
(1 \otimes \pi)\Delta(t) = t \otimes \pi(t)
\]

This \( t \) is defined up to a constant and determines uniquely the line bundle, hence the projective embedding of \( G/H \):

\[
\mathcal{O}(G/P)_n = \{ f \in \mathcal{O}(G) \mid (1 \otimes \pi)\Delta(f) = f \otimes \pi(t^n) \}
\]

**Example.** For \( G/P = G(2, 4) \) and the Plucker embedding, \( t = a_{11}a_{22} - a_{21}a_{12} \), \( \mathcal{O}(G/P) = \mathbb{C}[d_{ij}] \subset \mathbb{C}[G] \)
Let $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate.
Quantum Homogeneous varieties

Let $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate. $\mathbb{C}_q[G]$ is a quantum group if:
Quantum Homogeneous varieties

Let $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate. 
$\mathbb{C}_q[G]$ is a *quantum group* if:
Quantum Homogeneous varieties

Let $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate. $\mathbb{C}_q[G]$ is a quantum group if:

1. $\mathbb{C}_q[G]$ is a Hopf algebra over $\mathbb{C}_q$
Let $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate. $\mathbb{C}_q[G]$ is a quantum group if:

1. $\mathbb{C}_q[G]$ is a Hopf algebra over $\mathbb{C}_q$
2. $\mathbb{C}_q[G]$ is torsion-free, as a $\mathbb{C}_q$–module;
Quantum Homogeneous varieties

Let $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate. $
\mathbb{C}_q[G]$ is a quantum group if:

1. $\mathbb{C}_q[G]$ is a Hopf algebra over $\mathbb{C}_q$
2. $\mathbb{C}_q[G]$ is torsion-free, as a $\mathbb{C}_q$–module;
Quantum Homogeneous varieties

Let $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate. $\mathbb{C}_q[G]$ is a quantum group if:

1. $\mathbb{C}_q[G]$ is a Hopf algebra over $\mathbb{C}_q$
2. $\mathbb{C}_q[G]$ is torsion-free, as a $\mathbb{C}_q$–module;

The $\mathbb{C}_q$ algebra $\mathcal{O}_q(X)$ is a quantization of $\mathcal{O}(X)$ if

- it is torsion-free
Quantum Homogeneous varieties

Let $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate. $\mathbb{C}_q[G]$ is a quantum group if:

1. $\mathbb{C}_q[G]$ is a Hopf algebra over $\mathbb{C}_q$
2. $\mathbb{C}_q[G]$ is torsion-free, as a $\mathbb{C}_q$–module;

The $\mathbb{C}_q$ algebra $\mathcal{O}_q(X)$ is a quantization of $\mathcal{O}(X)$ if

- it is torsion-free
- $\mathcal{O}_q(X)/(q - 1)\mathcal{O}_q(X) \cong \mathcal{O}(X)$.

If $X$ is a projective variety, then we associate to it the $\mathbb{Z}$-graded algebra $\mathcal{O}(X)$ obtained through its embedding into $\mathbb{P}^m$. 
Quantum Homogeneous varieties

Let $C_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate. $C_q[G]$ is a quantum group if:

1. $C_q[G]$ is a Hopf algebra over $C_q$
2. $C_q[G]$ is torsion-free, as a $C_q$–module;

The $C_q$ algebra $O_q(X)$ is a quantization of $O(X)$ if
   - it is torsion-free
   - $O_q(X)/(q - 1)O_q(X) \cong O(X)$.

If $X$ is a projective variety, then we associate to it the $\mathbb{Z}$-graded algebra $O(X)$ obtained through its embedding into $\mathbb{P}^m$. If $X$ is an homogenous space for $G$, then $O(X)$ admits a natural coaction of $C[G]$. 
Quantum Homogeneous varieties

Let $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate. $\mathbb{C}_q[G]$ is a quantum group if:

1. $\mathbb{C}_q[G]$ is a Hopf algebra over $\mathbb{C}_q$
2. $\mathbb{C}_q[G]$ is torsion-free, as a $\mathbb{C}_q$–module;

The $\mathbb{C}_q$ algebra $\mathcal{O}_q(X)$ is a quantization of $\mathcal{O}(X)$ if

- it is torsion-free
- $\mathcal{O}_q(X)/(q - 1)\mathcal{O}_q(X) \cong \mathcal{O}(X)$.

If $X$ is a projective variety, then we associate to it the $\mathbb{Z}$-graded algebra $\mathcal{O}(X)$ obtained through its embedding into $\mathbb{P}^m$. If $X$ is an homogenous space for $G$, then $\mathcal{O}(X)$ admits a natural coaction of $\mathbb{C}[G]$. We say that $\mathcal{O}_q(X)$ is a quantum homogeneous variety, if $\mathcal{O}_q(X)$ admits a coaction of the quantum group $\mathbb{C}_q[G]$, reducing to the coaction of $\mathbb{C}[G]$ on $\mathcal{O}(X)$ when $q = 1$. 
We define *quantum section* of the line bundle $\mathcal{L}$ on $G/P$ given by $t$, any $d \in \mathbb{C}_q[G]$ such that
We define *quantum section* of the line bundle $\mathcal{L}$ on $G/P$ given by $t$, any $d \in \mathbb{C}_q[G]$ such that

$$\Delta(d) \in (d \otimes d + \mathbb{C}_q[G] \otimes I_q(P))$$
Quantum Line Bundles

We define quantum section of the line bundle $\mathcal{L}$ on $G/P$ given by $t$, any $d \in \mathbb{C}_q[G]$ such that

\begin{enumerate}
\item $\Delta(d) \in (d \otimes d + \mathbb{C}_q[G] \otimes l_q(P))$
\item $d \mod (q-1) \mathbb{C}_q[G] = t$ \quad ($\in \mathbb{C}[G]$)
\end{enumerate}
Quantum Line Bundles

We define *quantum section* of the line bundle $\mathcal{L}$ on $G/P$ given by $t$, any $d \in \mathbb{C}_q[G]$ such that

1. $\Delta(d) \in (d \otimes d + \mathbb{C}_q[G] \otimes I_q(P))$
2. $d \mod (q-1) \mathbb{C}_q[G] = t$ ($\in \mathbb{C}[G]$)

where $\pi : \mathbb{C}_q[G] \longrightarrow \mathbb{C}_q[P] := \mathbb{C}_q[G]/I_q(P)$.
We define *quantum section* of the line bundle $\mathcal{L}$ on $G/P$ given by $t$, any $d \in \mathbb{C}_q[G]$ such that

1. $\Delta(d) \in (d \otimes d + \mathbb{C}_q[G] \otimes I_q(P))$
2. $d \mod (q-1) \mathbb{C}_q[G] = t \quad (\in \mathbb{C}[G])$

where $\pi : \mathbb{C}_q[G] \longrightarrow \mathbb{C}_q[P] := \mathbb{C}_q[G]/I_q(P)$.

A quantum section is a quantization of a line bundle on $G/P$, hence of a projective embedding of $G/P$. 

Rita Fioresi
Quantum flags, quantum line bundles and the Quantum Duality Principle
September 18, 2019 7 / 21
Quantum Line Bundles

We define *quantum section* of the line bundle $\mathcal{L}$ on $G/P$ given by $t$, any $d \in \mathbb{C}_q[G]$ such that

1. $\Delta(d) \in (d \otimes d + \mathbb{C}_q[G] \otimes I_q(P))$
2. $d \mod (q-1)\mathbb{C}_q[G] = t \quad (\in \mathbb{C}[G])$

where $\pi : \mathbb{C}_q[G] \longrightarrow \mathbb{C}_q[P] := \mathbb{C}_q[G]/I_q(P)$.

A *quantum section* is a quantization of a line bundle on $G/P$, hence of a projective embedding of $G/P$.

Define:

$$\mathcal{O}_q(G/P) := \sum \mathcal{O}_q(G/P)_n, \quad \text{where}$$

$$\mathcal{O}_q(G/P)_n := \{ f \in \mathbb{C}_q[G] \mid (id \otimes \pi)\Delta(f) = f \otimes \pi(d^n) \}.$$
Quantum Section and Projective homogeneous spaces
Theorem (Ciccoli, F., Gavarini)
Theorem (Ciccoli, F., Gavarini)
Let \( d \) be a quantum section on \( G/P \). Then
Theorem (Ciccoli, F., Gavarini)
Let $d$ be a quantum section on $G/P$. Then

$$\mathcal{O}_q(G/P)_r \cdot \mathcal{O}_q(G/P)_s \subseteq \mathcal{O}_q(G/P)_{r+s} \subset \mathbb{C}_q[G]$$

Hence, $\mathcal{O}_q(G/P)$ is a graded subalgebra.
Theorem (Ciccoli, F., Gavarini)

Let $d$ be a quantum section on $G/P$. Then

$$\mathcal{O}_q(G/P)_r \cdot \mathcal{O}_q(G/P)_s \subseteq \mathcal{O}_q(G/P)_{r+s} \subset \mathbb{C}_q[G]$$

Hence, $\mathcal{O}_q(G/P)$ is a graded subalgebra. and we have:

$$\mathcal{O}_q(G/P) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}_q(G/P)_n \subset \mathbb{C}_q[G].$$
Theorem (Ciccoli, F., Gavarini)
Let \( d \) be a quantum section on \( G/P \). Then

1. \[
\mathcal{O}_q(G/P)_r \cdot \mathcal{O}_q(G/P)_s \subseteq \mathcal{O}_q(G/P)_{r+s} \subset \mathbb{C}_q[G]
\]
Hence, \( \mathcal{O}_q(G/P) \) is a graded subalgebra. and we have:
\[
\mathcal{O}_q(G/P) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}_q(G/P)_n \subset \mathbb{C}_q[G].
\]

2. \( \mathcal{O}_q(G/P) \) is a graded \( \mathbb{C}_q[G] \)-comodule
\[
\Delta|_{\mathcal{O}_q(G/P)} : \mathcal{O}_q(G/P) \longrightarrow \mathbb{C}_q[G] \otimes \mathcal{O}_q(G/P)
\]
Theorem (Ciccoli, F., Gavarini)

Let \( d \) be a quantum section on \( G/P \). Then

1. 
\[ \mathcal{O}_q(G/P)_r \cdot \mathcal{O}_q(G/P)_s \subseteq \mathcal{O}_q(G/P)_{r+s} \subset \mathbb{C}_q[G] \]

Hence, \( \mathcal{O}_q(G/P) \) is a graded subalgebra. and we have:

\[ \mathcal{O}_q(G/P) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}_q(G/P)_n \subset \mathbb{C}_q[G]. \]

2. \( \mathcal{O}_q(G/P) \) is a graded \( \mathbb{C}_q[G] \)-comodule

\[ \Delta|_{\mathcal{O}_q(G/P)} : \mathcal{O}_q(G/P) \longrightarrow \mathbb{C}_q[G] \otimes \mathcal{O}_q(G/P) \]

3. We have

\[ \mathcal{O}_q(G/P) \cap (q-1)\mathbb{C}_q[G] = (q-1)\mathcal{O}_q(G/P) \]
Theorem (Ciccoli, F., Gavarini)
Let \( d \) be a quantum section on \( G/P \). Then

1. \( \mathcal{O}_q(G/P)_r \cdot \mathcal{O}_q(G/P)_s \subseteq \mathcal{O}_q(G/P)_{r+s} \subset \mathbb{C}_q[G] \)

Hence, \( \mathcal{O}_q(G/P) \) is a graded subalgebra. and we have:

\[ \mathcal{O}_q(G/P) = \bigoplus_{n \in \mathbb{N}} \mathcal{O}_q(G/P)_n \subset \mathbb{C}_q[G]. \]

2. \( \mathcal{O}_q(G/P) \) is a graded \( \mathbb{C}_q[G] \)-comodule

\[ \Delta|_{\mathcal{O}_q(G/P)} : \mathcal{O}_q(G/P) \longrightarrow \mathbb{C}_q[G] \otimes \mathcal{O}_q(G/P) \]

3. We have

\[ \mathcal{O}_q(G/P) \cap (q-1) \mathbb{C}_q[G] = (q-1) \mathcal{O}_q(G/P) \]

\( \mathcal{O}_q(G/P) \) is a projective homogeneous quantum variety.
Quantum Special Linear group

Define the *quantum matrices*

\[
\mathbb{C}_q[M_n] = \mathbb{C}_q\langle a_{ij}\rangle / I_M
\]  

(1)

where \( I_M \) is the ideal of the Manin relations:
Quantum Special Linear group

Define the *quantum matrices*

\[ \mathbb{C}_q[M_n] = \mathbb{C}_q \langle a_{ij} \rangle / I_M \]  

(1)

where \( I_M \) is the ideal of the Manin relations:

\[
\begin{align*}
    a_{ij}a_{kj} &= q^{-1}a_{kj}a_{ij} & i < k \\
    a_{ij}a_{kl} &= a_{kl}a_{ij} & i < k, j > l \text{ or } i > k, j < l \\
    a_{ij}a_{il} &= q^{-1}a_{il}a_{ij} & j < l \\
    a_{ij}a_{kl} - a_{kl}a_{ij} &= (q^{-1} - q)a_{ik}a_{jl} & i < k, j < l
\end{align*}
\]

The quantum matrix algebra \( \mathcal{O}_q(M_n) \) is a bialgebra, with:

\[
\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}, \quad \epsilon(a_{ij}) = \delta_{ij}.
\]

Define *quantum special linear group* to be the algebra
Quantum Special Linear group

Define the quantum matrices

\[ \mathbb{C}_q[M_n] = \mathbb{C}_q\langle a_{ij} \rangle / I_M \]  

(1)

where \( I_M \) is the ideal of the Manin relations:

\[ a_{ij}a_{kj} = q^{-1}a_{kj}a_{ij} \quad i < k \quad a_{ij}a_{kl} = a_{kl}a_{ij} \quad i < k, j > l \quad \text{or} \quad i > k, j < l \]

\[ a_{ij}a_{il} = q^{-1}a_{il}a_{ij} \quad j < l \quad a_{ij}a_{kl} - a_{kl}a_{ij} = (q^{-1} - q)a_{ik}a_{jl} \quad i < k, j < l \]

The quantum matrix algebra \( \mathcal{O}_q(M_n) \) is a bialgebra, with:

\[ \Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}, \quad \epsilon(a_{ij}) = \delta_{ij}. \]

Define quantum special linear group to be the algebra

\[ \mathbb{C}_q[\text{SL}_n] = \mathbb{C}_q[M]/(\det_q - 1) \]
The quantum Projective space

Let $P \subset SL_n(\mathbb{C})$: 
The quantum Projective space

Let $P \subset \text{SL}_n(\mathbb{C})$:

$$
P = \left\{ \begin{pmatrix} t_{11} & p_{12} & \cdots & p_{1n} \\ 0 & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & s_{n2} & \cdots & s_{nn} \end{pmatrix} \right\} \subset G = \left\{ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad \det(A) = 1 \right\}
$$

In this case $G/P \simeq \mathbb{P}^{n-1}$ and
The quantum Projective space

Let $P \subset SL_n(\mathbb{C})$:

$$P = \left\{ \begin{pmatrix} t_{11} & p_{12} & \cdots & p_{1n} \\ 0 & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & s_{n2} & \cdots & s_{nn} \end{pmatrix} \right\} \subset G = \left\{ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mid \det(A) = 1 \right\}$$

In this case $G/P \simeq \mathbb{P}^{n-1}$ and

$$O_q(\mathbb{P}^{n-1}) = \mathbb{C}[x_0, \ldots, x_{n-1}]/(x_i x_j - q^{-1} x_i x_j, i < j)$$
The quantum Projective space

Let $P \subset SL_n(\mathbb{C})$:

$$P = \left\{ \begin{pmatrix} t_{11} & p_{12} & \ldots & p_{1n} \\ 0 & s_{22} & \ldots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & s_{n2} & \ldots & s_{nn} \end{pmatrix} \right\} \subset G = \left\{ A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \mid \det(A) = 1 \right\}$$

In this case $G/P \simeq \mathbb{P}^{n-1}$ and

$$O_q(\mathbb{P}^{n-1}) = \mathbb{C}[x_0, \ldots, x_{n-1}]/(x_i x_j - q^{-1} x_i x_j, i < j)$$

$d = a_{11} \in O_q(G)$ is a quantum section, $d_i = a_{i,1}$, in fact:
The quantum Projective space

Let $P \subset \text{SL}_n(\mathbb{C})$:

$$P = \left\{ \begin{pmatrix} t_{11} & p_{12} & \ldots & p_{1n} \\ 0 & s_{22} & \ldots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & s_{n2} & \ldots & s_{nn} \end{pmatrix} \right\} \subset G = \left\{ A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \ \text{det}(A) = 1 \right\}$$

In this case $G/P \cong \mathbb{P}^{n-1}$ and

$$\mathcal{O}_q(\mathbb{P}^{n-1}) = \mathbb{C}[x_0, \ldots, x_{n-1}]/(x_ix_j - q^{-1}x_ix_j, i < j)$$

$d = a_{11} \in \mathcal{O}_q(G)$ is a quantum section, $d_i = a_{i,1}$, in fact:

$$(1 \otimes \pi)\Delta(a_{11}) = a_{11} \otimes \pi(a_{11}), \quad \Delta(a_{11}) = \sum a_{1j} \otimes a_{j1}$$
The quantum Projective space

Let \( P \subset \text{SL}_n(\mathbb{C}) \):

\[
P = \left\{ \begin{pmatrix} t_{11} & p_{12} & \cdots & p_{1n} \\ 0 & s_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & s_{n2} & \cdots & s_{nn} \end{pmatrix} \right\} \subset G = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \right\} \quad \text{det}(A) = 1
\]

In this case \( G/P \simeq \mathbb{P}^{n-1} \) and

\[
\mathcal{O}_q(\mathbb{P}^{n-1}) = \mathbb{C}[x_0, \ldots, x_{n-1}]/(x_i x_j - q^{-1} x_i x_j, i < j)
\]

\( d = a_{11} \in \mathcal{O}_q(G) \) is a quantum section, \( d_i = a_{i,1} \), in fact:

\[
(1 \otimes \pi)\Delta(a_{11}) = a_{11} \otimes \pi(a_{11}), \quad \Delta(a_{11}) = \sum a_{1j} \otimes a_{j1}
\]

Hence \( \mathcal{O}_q(G/P)_1 = \mathcal{O}_q(\mathbb{P}^{n-1})_1 \) is the span of the \( x_i = a_{i,1} \),

\[
\pi : \mathbb{C}_q[G] \longrightarrow \mathbb{C}_q[P].
\]
The Plucker embedding of the Quantum Grassmannian
Key Observations.

- $d_{12} \in \mathbb{C}_q[SL_4]$ is a quantum section, in fact:
Key Observations.

- $d_{12} \in \mathbb{C}_q[SL_4]$ is a quantum section, in fact:

\[
(i \otimes \pi)\Delta(d_{12}) = (i \otimes \pi)( \sum_{1 \leq i < j \leq 4} d_{12}^{ij} \otimes d_{12}^{12} ) = d_{12} \otimes d_{12}
\]

where $d_{colindices \ rowindices}^{colindices \ rowindices}$. 

Rita Fioresi
Quantum flags, quantum line bundles and the Quantum Duality Principle

September 18, 2019 11 / 21
Key Observations.

- $d_{12} \in \mathbb{C}_q[SL_4]$ is a quantum section, in fact:
  \[
  (id \otimes \pi) \Delta(d_{12}) = (id \otimes \pi)( \sum_{1 \leq i < j \leq 4} d_{12}^{ij} \otimes d_{12}^{12} ) = d_{12} \otimes d_{12}
  \]
  where $d_{\text{col indeces}}^{\text{row indices}}$.

- $d_{ij}$ generate $\mathcal{O}_q(\text{Gr})$!
Key Observations.

- \( d_{12} \in \mathbb{C}_q[SL_4] \) is a quantum section, in fact:

\[
(id \otimes \pi) \Delta (d_{12}) = (id \otimes \pi) \left( \sum_{1 \leq i < j \leq 4} d_{1j}^{ij} \otimes d_{12}^{ij} \right) = d_{12} \otimes d_{12}
\]

where \( d_{\text{col indeces}} \) \( d_{\text{row indeces}} \).

- \( d_{ij} \) generate \( \mathcal{O}_q(\text{Gr}) \)!

Proposition.
The Plucker embedding of the Quantum Grassmannian

Key Observations.
- $d_{12} \in \mathbb{C}_q[SL_4]$ is a quantum section, in fact:
  \[(\text{id} \otimes \pi) \Delta(d_{12}) = (\text{id} \otimes \pi)( \sum_{1 \leq i < j \leq 4} d_{12}^{ij} \otimes d_{12}^{ij}) = d_{12} \otimes d_{12}\]
  where $d_{\text{col indeces}} \otimes d_{\text{row indices}}$.
- $d_{ij}$ generate $\mathcal{O}_q(\text{Gr})$!

Proposition.

\[
\mathcal{O}_q(\text{Gr}) = \frac{\mathbb{C}[d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}]}{(d_{12}d_{34} - q^{-1}d_{13}d_{24} + q^{-2}d_{14}d_{23}, l_{\text{comm}})}
\]

is a quantization of the homogeneous ring of the grassmannian with respect to the Plucker embedding.
Key Observations.

- $d_{12} \in \mathbb{C}_q[SL_4]$ is a quantum section, in fact:
  \[
  (id \otimes \pi)\Delta(d_{12}) = (id \otimes \pi)(\sum_{1 \leq i < j \leq 4} d_{12}^{ij} \otimes d_{12}^{ij}) = d_{12} \otimes d_{12}
  \]
  where $d_{ij}$ generate $O_q(Gr)$!

Proposition.

\[
O_q(Gr) = \frac{\mathbb{C}[d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}]}{(d_{12}d_{34} - q^{-1}d_{13}d_{24} + q^{-2}d_{14}d_{23}, I_{\text{comm}})}
\]

is a quantization of the homogeneous ring of the grassmannian with respect to the Plucker embedding. It carries a natural coaction of the quantum special linear group.

\[
\Delta(d_{ij}) = \sum_{1 \leq i < j \leq 4} d_{12}^{ij} \otimes d_{12}^{ij}
\]

The generalization to $n$ dimensions is immediate!
Projective embeddings of Quantum flags
Facts
Facts

- $a_{11}d_{12}d_{123} \ldots$ is a quantum section.
Facts

- $a_{11}d_{12}d_{123} \ldots$ is a quantum section.
- $d_I$ the determinants of the quantum minors generate $\mathcal{O}_q(\text{Flag})$
Facts

- $a_{11}d_{12}d_{123}\ldots$ is a quantum section.
- $d_I$ the determinants of the quantum minors generate $\mathcal{O}_q(\text{Flag})$.
- Commutation, Incidence and Plucker relations.
Projective embeddings of Quantum flags

Facts

- $a_{11}d_{12}d_{123} \ldots$ is a quantum section.
- $d_I$ the determinants of the quantum minors generate $\mathcal{O}_q(\text{Flag})$
- Commutation, Incidence and Plucker relations.

Proposition.
Facts

- $a_{11}d_{12} d_{123} \ldots$ is a quantum section.
- $d_I$ the determinants of the quantum minors generate $\mathcal{O}_q(\text{Flag})$.
- Commutation, Incidence and Plucker relations.

Proposition.

$$\mathcal{O}_q(\text{Flag}) = \frac{\mathbb{C}[d_i, d_{ij}, d_{ijk}, \ldots]}{(l_{\text{incidence}}, l_{\text{Plucker}}, l_{\text{comm}})}$$

is a quantization of the homogeneous ring of the flag with respect to the chosen projective embedding.
Facts

- $a_{11}d_{12}d_{123}\ldots$ is a quantum section.
- $d_I$ the determinants of the quantum minors generate $\mathcal{O}_q(\text{Flag})$.
- Commutation, Incidence and Plucker relations.

Proposition.

\[ \mathcal{O}_q(\text{Flag}) = \frac{\mathbb{C}[d_i, d_{ij}, d_{ijk}, \ldots]}{(l_{\text{incidence}}, l_{\text{Plucker}}, l_{\text{comm}})} \]

is a quantization of the homogeneous ring of the flag with respect to the chosen projective embedding. It carries a natural coaction of the quantum special linear group.
Hypotheses
Hypotheses

- $G$ Poisson algebraic group;
Hypotheses

- $G$ Poisson algebraic group;
- $K < G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
The Quantum Duality Principle: Setting

Hypotheses

- $G$ Poisson algebraic group;
- $K \triangleleft G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
- $K$ coisotropic subgroup i.e. equivalently:
The Quantum Duality Principle: Setting

Hypotheses

- $G$ Poisson algebraic group;
- $K < G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
- $K$ coisotropic subgroup i.e. equivalently:
  - $\mathcal{I}(K)$ is a Poisson subalgebra of $\mathbb{C}[G]$
Hypotheses

- $G$ Poisson algebraic group;
- $K < G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
- $K$ coisotropic subgroup i.e. equivalently:
  1. $\mathcal{I}(K)$ is a Poisson subalgebra of $\mathbb{C}[G]$
  2. $\mathcal{U}(\mathfrak{k})$ is a Poisson coideal of $\mathcal{U}(\mathfrak{g})$
Hypotheses

- $G$ Poisson algebraic group;
- $K < G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
- $K$ coisotropic subgroup i.e. equivalently:
  1. $\mathcal{I}(K)$ is a Poisson subalgebra of $\mathbb{C}[G]$
  2. $\mathcal{U}(\mathfrak{k})$ is a Poisson coideal of $\mathcal{U}(\mathfrak{g})$
  3. $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$.
The Quantum Duality Principle: Setting

Hypotheses

- $G$ Poisson algebraic group;
- $K < G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
- $K$ coisotropic subgroup i.e. equivalently:
  1. $\mathcal{I}(K)$ is a Poisson subalgebra of $\mathbb{C}[G]$
  2. $\mathcal{U}(\mathfrak{k})$ is a Poisson coideal of $\mathcal{U}(\mathfrak{g})$
  3. $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$.

Proposition. The following conditions are equivalent:
The Quantum Duality Principle: Setting

Hypotheses

- $G$ Poisson algebraic group;
- $K < G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
- $K$ coisotropic subgroup i.e. equivalently:
  1. $\mathcal{I}(K)$ is a Poisson subalgebra of $\mathbb{C}[G]$
  2. $\mathcal{U}(\mathfrak{k})$ is a Poisson coideal of $\mathcal{U}(\mathfrak{g})$
  3. $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$.

Proposition. The following conditions are equivalent:

- $M = G/K$ is a Poisson variety;
Hypotheses

- $G$ Poisson algebraic group;
- $K \triangleleft G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
- $K$ coisotropic subgroup i.e. equivalently:
  1. $\mathcal{I}(K)$ is a Poisson subalgebra of $\mathbb{C}[G]$
  2. $\mathcal{U}(\mathfrak{k})$ is a Poisson coideal of $\mathcal{U}(\mathfrak{g})$
  3. $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$.

Proposition. The following conditions are equivalent:

1. $M = G/K$ is a Poisson variety;
2. $\mathbb{C}[G]^K$ is a Poisson subalgebra of $\mathbb{C}[G]$;
Hypotheses

- $G$ Poisson algebraic group;
- $K < G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
- $K$ coisotropic subgroup i.e. equivalently:
  1. $\mathcal{I}(K)$ is a Poisson subalgebra of $\mathbb{C}[G]$ 
  2. $\mathcal{U}(\mathfrak{k})$ is a Poisson coideal of $\mathcal{U}(\mathfrak{g})$
  3. $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$.

Proposition. The following conditions are equivalent:

- $M = G/K$ is a Poisson variety;
- $\mathbb{C}[G]^K$ is a Poisson subalgebra of $\mathbb{C}[G]$;
- $\mathcal{U}(\mathfrak{g})\mathfrak{k}$ is a Poisson coideal of $\mathcal{U}(\mathfrak{g})$
Hypotheses

- $G$ Poisson algebraic group;
- $K < G$ subgroup, with $\mathcal{I}(K)$ ideal of $K$ in $\mathbb{C}[G]$;
- $K$ coisotropic subgroup i.e. equivalently:
  1. $\mathcal{I}(K)$ is a Poisson subalgebra of $\mathbb{C}[G]$;
  2. $\mathcal{U}(\mathfrak{k})$ is a Poisson coideal of $\mathcal{U}(\mathfrak{g})$;
  3. $\mathfrak{k}^\perp$ is a Lie subalgebra of $\mathfrak{g}^*$.

Proposition. The following conditions are equivalent:

1. $M = G/K$ is a Poisson variety;
2. $\mathbb{C}[G]^K$ is a Poisson subalgebra of $\mathbb{C}[G]$;
3. $\mathcal{U}(\mathfrak{g})\mathfrak{k}$ is a Poisson coideal of $\mathcal{U}(\mathfrak{g})$;
4. $K$ is a coisotropic subgroup of $G$. 
Groups. $G$ Poisson Group $\Rightarrow \mathfrak{g}$ Lie bialgebra $\Rightarrow \mathfrak{g}^*$ Lie bialgebra $\Rightarrow$ There exists a Poisson group $G^*$. 
**Groups.** $G$ Poisson Group $\implies \mathfrak{g}$ Lie bialgebra $\implies \mathfrak{g}^*$ Lie bialgebra $\implies$ There exists a Poisson group $G^*$.

**Subgroups.** $K$ isotropic subgroup of $G$ $\implies \mathfrak{k}^\perp$ Lie subalgebra of $\mathfrak{g}^*$ $\implies$ There exists a coisotropic subgroup $K^\perp$ of $G^*$. 
**Groups.** $G$ Poisson Group $\implies \mathfrak{g}$ Lie bialgebra $\implies \mathfrak{g}^*$ Lie bialgebra $\implies$ There exists a Poisson group $G^*$.

**Subgroups.** $K$ isotropic subgroup of $G$ $\implies \mathfrak{k}^\perp$ Lie subalgebra of $\mathfrak{g}^*$ $\implies$ There exists a coisotropic subgroup $K^\perp$ of $G^*$.

**Groups.** $M = G/K$ Poisson quotient $\implies M^\times = G^*/K^\perp$ is a Poisson quotient.
Poisson Duality

- **Groups.** $G$ Poisson Group $\implies \mathfrak{g}$ Lie bialgebra $\implies \mathfrak{g}^*$ Lie bialgebra $\implies$ There exists a Poisson group $G^*$.

- **Subgroups.** $K$ isotropic subgroup of $G$ $\implies \mathfrak{k}^\perp$ Lie subalgebra of $\mathfrak{g}^*$ $\implies$ There exists a coisotropic subgroup $K^\perp$ of $G^*$.

- **Groups.** $M = G/K$ Poisson quotient $\implies M^\times = G^*/K^\perp$ is a Poisson quotient.

**Example:** Stokes matrices (Dubrovin, Ugaglia).
**Groups.** \(G\) Poisson Group \(\Rightarrow \mathfrak{g}\) Lie bialgebra \(\Rightarrow \mathfrak{g}^*\) Lie bialgebra \(\Rightarrow\) There exists a Poisson group \(G^*\).

**Subgroups.** \(K\) isotropic subgroup of \(G\) \(\Rightarrow\) \(\mathfrak{k}^\perp\) Lie subalgebra of \(\mathfrak{g}^*\) \(\Rightarrow\) There exists a coisotropic subgroup \(K^\perp\) of \(G^*\).

**Groups.** \(M = G/K\) Poisson quotient \(\Rightarrow M^\times = G^*/K^\perp\) is a Poisson quotient.

**Example:** Stokes matrices (Dubrovin, Ugaglia).
\(K = \text{SO}(n) \subset G = \text{SL}_n, M = \text{SL}_n/\text{SO}(n)\).
**Groups.** $G$ Poisson Group $\implies \mathfrak{g}$ Lie bialgebra $\implies \mathfrak{g}^*$ Lie bialgebra $\implies$ There exists a Poisson group $G^*$.

**Subgroups.** $K$ isotropic subgroup of $G$ $\implies \mathfrak{k}^\perp$ Lie subalgebra of $\mathfrak{g}^*$ $\implies$ There exists a coisotropic subgroup $K^\perp$ of $G^*$.

**Groups.** $M = G/K$ Poisson quotient $\implies M^\times = G^*/K^\perp$ is a Poisson quotient.

**Example:** Stokes matrices (Dubrovin, Ugaglia).

$K = \text{SO}(n) \subset G = \text{SL}_n$, $M = \text{SL}_n/\text{SO}(n)$.

$\mathfrak{g}^* = \{(X, Y) \in \mathfrak{b}^+ \oplus \mathfrak{b}^- | X|_\hbar = -Y|_\hbar\}$, $G^* = B_+ \star B_-$. pairs of upper/lower triangular matrices with mutually inverse diagonal.
Groups. $G$ Poisson Group $\Rightarrow \mathfrak{g}$ Lie bialgebra $\Rightarrow \mathfrak{g}^*$ Lie bialgebra $\Rightarrow$ There exists a Poisson group $G^*$.

Subgroups. $K$ isotropic subgroup of $G$ $\Rightarrow \mathfrak{k}^\perp$ Lie subalgebra of $\mathfrak{g}^*$ $\Rightarrow$ There exists a coisotropic subgroup $K^\perp$ of $G^*$.

Groups. $M = G/K$ Poisson quotient $\Rightarrow M^\times = G^*/K^\perp$ is a Poisson quotient.

Example: Stokes matrices (Dubrovin, Ugaglia).

$K = \text{SO}(n) \subset G = \text{SL}_n$, $M = \text{SL}_n/\text{SO}(n)$.

$\mathfrak{g}^* = \{ (X, Y) \in \mathfrak{b}^+ \oplus \mathfrak{b}^- \mid X|_{\mathfrak{h}} = -Y|_{\mathfrak{h}} \}$, $G^* = B_+ \star B_-$. pairs of upper/lower triangular matrices with mutually inverse diagonal.

$\text{SO}(n)^\perp = \{ (B, C) \mid BC^t = I \} \subset \text{SL}_n^*$. 
Groups. \( G \) Poisson Group \( \rightarrow \mathfrak{g} \) Lie bialgebra \( \rightarrow \mathfrak{g}^* \) Lie bialgebra \( \rightarrow \) There exists a Poisson group \( G^* \).

Subgroups. \( K \) isotropic subgroup of \( G \) \( \rightarrow \mathfrak{k}^\perp \) Lie subalgebra of \( \mathfrak{g}^* \) \( \rightarrow \) There exists a coisotropic subgroup \( K^\perp \) of \( G^* \).

Groups. \( M = G/K \) Poisson quotient \( \rightarrow M^\times = G^*/K^\perp \) is a Poisson quotient.

Example: Stokes matrices (Dubrovin, Ugaglia).
\( K = \text{SO}(n) \subset G = \text{SL}_n, \ M = \text{SL}_n/\text{SO}(n). \)
\( \mathfrak{g}^* = \{(X, Y) \in \mathfrak{b}^+ \oplus \mathfrak{b}^- \mid X|_h = -Y|_h\}, \ G^* = B_+ \ast B_- \). pairs of upper/lower triangular matrices with mutually inverse diagonal.
\( \text{SO}(n)^\perp = \{(B, C) \mid BC^t = I\} \subset \text{SL}^*_n. \)
\( M^\times = \text{SL}^*_n/\text{SO}(n)^\perp \) are the Stokes matrices and QDP provides a quantization for them.
The Quantum Duality Principle

(Drinfeld 1986, Gavarini 2002): There is a functorial correspondence

\[ O_q(G) \hookrightarrow \mathcal{U}_h(g^*), \quad \mathcal{U}_h(g) \hookrightarrow O_q(G^*) \]

This correspondence suitably restricts:
(Drinfeld 1986, Gavarini 2002): There is a functorial correspondence
\[ O_q(G) \mapsto U_h(g^*), \quad U_h(g) \mapsto O_q(G^*) \]
This correspondence suitably restricts:
\[ I_q(K) \mapsto I_h(K) = \left( U(g^*) \mathfrak{k} \right)_h \]
(Drinfeld 1986, Gavarini 2002): There is a functorial correspondence
\[ \mathcal{O}_q(G) \mapsto \mathcal{U}_h(g^*), \quad U_h(g) \mapsto \mathcal{O}_q(G^*) \]

This correspondence suitably restricts:
\[ \mathcal{I}_q(K) \mapsto \mathcal{I}_h(K)^\vee = \left( U\left(g^*\right) \mathfrak{k}^\perp \right)_h \]
\[ \mathbb{C}_q[G]^K \mapsto \left( \mathbb{C}[G]^K \right)^\vee_h = U_h(\mathfrak{k}^\perp) \]
(Drinfeld 1986, Gavarini 2002): There is a functorial correspondence

\[ \mathcal{O}_q(G) \leftrightarrow U_h(\mathfrak{g}^*), \quad U_h(\mathfrak{g}) \leftrightarrow \mathcal{O}_q(G^*) \]

This correspondence suitably restricts:

\[ I_q(K) \leftrightarrow I_h(K)^\vee = \left( U(\mathfrak{g}^*) \mathfrak{k}^\perp \right)_h \]
\[ \mathbb{C}_q[G]^K \leftrightarrow \left( \mathbb{C}[G]^K \right)_h^\vee = U_h(\mathfrak{k}^\perp) \]
\[ U_h(\mathfrak{k}) \leftrightarrow U_h(\mathfrak{k})' = \mathbb{C}_q[G^*]^K^\perp \]
(Drinfeld 1986, Gavarini 2002): There is a functorial correspondence

\[ \mathcal{O}_q(G) \leftrightarrow U_h(g^*), \quad U_h(g) \mapsto \mathcal{O}_q(G^*) \]

This correspondence suitably restricts:

\[ I_q(K) \mapsto I_h(K)^\vee = \left( U(g^*)^\perp \right)_h \]
\[ \mathbb{C}_q[G]^K \mapsto \left( \mathbb{C}[G]^K \right)^\vee_h = U_h(\mathfrak{k}^\perp) \]
\[ U_h(\mathfrak{k}) \mapsto U_h(\mathfrak{k})' = \mathbb{C}_q[G^*]^{K^\perp} \]
\[ U_h(g) \mathfrak{k} \mapsto \left( U_h(g) \mathfrak{k} \right)' = I_h(K^\perp) \]
The Quantum Duality Principle: Projective Homogeneous Spaces

QDP recipe:
The Quantum Duality Principle: Projective Homogeneous Spaces

QDP recipe:

\[ \mathcal{O}_q(G)^\vee := \sum (q - 1)^{-n} I_G^n, \quad I_G = \ker \epsilon + (q - 1) \mathcal{O}_q(G) \]
QDP recipe:

\[ \mathcal{O}_q(G)^\vee := \sum (q-1)^{-n} l^n_G, \quad l_G = \ker \epsilon + (q-1) \mathcal{O}_q(G) \]

\[ \mathcal{O}_q(G/K)^\vee := \sum (q-1)^{-n} l^n_{G/K}, \quad l_{G/K} = \ker \epsilon|_{\mathcal{O}_q(G/K)} + (q-1) \mathcal{O}_q(G) \]
The Quantum Duality Principle: Projective Homogeneous Spaces

QDP recipe:

\[ \mathcal{O}_q(G)^\vee := \sum (q-1)^{-n} l_G^n, \quad l_G = \ker \epsilon + (q-1)\mathcal{O}_q(G) \]

\[ \mathcal{O}_q(G/K)^\vee := \sum (q-1)^{-n} l_{G/K}^n, \quad l_{G/K} = \ker \epsilon|_{\mathcal{O}_q(G/K)} + (q-1)\mathcal{O}_q(G) \]

Theorem (Ciccoli-F.-Gavarini).

Rita Fioresi  
Quantum flags, quantum line bundles and the Quantum Duality Principle  
September 18, 2019 16 / 21
The Quantum Duality Principle: Projective Homogeneous Spaces

QDP recipe:

\[ \mathcal{O}_q(G)^\vee := \sum (q - 1)^{-n} l^n_G, \quad l_G = \ker \epsilon + (q - 1)\mathcal{O}_q(G) \]

\[ \mathcal{O}_q(G/K)^\vee := \sum (q - 1)^{-n} l^n_{G/K}, \quad l_{G/K} = \ker \epsilon|_{\mathcal{O}_q(G/K)} + (q - 1)\mathcal{O}_q(G) \]

Theorem (Ciccoli-F.-Gavarini).

- \( \mathcal{O}_q(G/K)^\vee \) is a quantization of \( \mathcal{U}(\mathfrak{k}^\perp) \) as a subalgebra of \( \mathcal{U}(\mathfrak{g}^*) \).
The Quantum Duality Principle: Projective Homogeneous Spaces

QDP recipe:

\[
O_q(G)^\vee := \sum (q - 1)^{-n} l^n_G, \quad l_G = \ker \epsilon + (q - 1)O_q(G)
\]

\[
O_q(G/K)^\vee := \sum (q - 1)^{-n} l^n_{G/K}, \quad l_{G/K} = \ker \epsilon|_{O_q(G/K)} + (q - 1)O_q(G)
\]

**Theorem (Ciccoli-F.-Gavarini).**

- \(O_q(G/K)^\vee\) is a quantization of \(\mathcal{U}(\mathfrak{t}^\perp)\) as a subalgebra of \(\mathcal{U}(\mathfrak{g}^*)\).
The Quantum Duality Principle: Projective Homogeneous Spaces

QDP recipe:

\[ \mathcal{O}_q(G)^\vee := \sum (q-1)^{-n} l_G^n, \quad I_G = \ker \epsilon + (q-1)\mathcal{O}_q(G) \]

\[ \mathcal{O}_q(G/K)^\vee := \sum (q-1)^{-n} l_{G/K}^n, \quad I_{G/K} = \ker \epsilon |_{\mathcal{O}_q(G/K)} + (q-1)\mathcal{O}_q(G) \]

**Theorem (Ciccoli-F.-Gavarini).**

- \( \mathcal{O}_q(G/K)^\vee \) is a quantization of \( \mathcal{U}(\mathfrak{t}^\perp) \) as a subalgebra of \( \mathcal{U}(\mathfrak{g}^*) \).
- \( \hat{\mathcal{O}}_q(G/K)^\vee \) is a quantization of \( \mathcal{U}(\mathfrak{t}^\perp) \) as a subalgebra and left ideal of \( \mathcal{U}(\mathfrak{g}^*) \). In other words, it is an infinitesimal deformation of the coisotropic subgroup \( K^\perp \) of \( G \). Furthermore it carries a natural coaction of \( \hat{\mathcal{O}}_q(G)^\vee \), so it is a (infinitesimal) quantum homogeneous space.
$\mathfrak{gl}_n$ is a Lie bialgebra, its dual space $\mathfrak{gl}_n^* \subset \mathfrak{gl}_n \oplus \mathfrak{gl}_n$:
$\mathfrak{gl}_n$ is a Lie bialgebra, its dual space $\mathfrak{gl}_n^* \subset \mathfrak{gl}_n \oplus \mathfrak{gl}_n$:

\[
\begin{pmatrix}
-m_{11} & 0 & \cdots & 0 \\
m_{21} & -m_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
m_{n-1,1} & m_{n-1,2} & \cdots & 0 \\
m_{n,1} & m_{n,2} & \cdots & -m_{n,n}
\end{pmatrix}, \begin{pmatrix}
m_{11} & m_{12} & \cdots & m_{1,n-1} & m_{1,n} \\
0 & m_{22} & \cdots & m_{2,n-1} & m_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & m_{n-1,n-1} & m_{n-1,n} \\
0 & 0 & \cdots & 0 & m_{n,n}
\end{pmatrix}
\]
gl\_n is a Lie bialgebra, its dual space \( gl\_n^* \subset gl\_n \oplus gl\_n \):

\[
\begin{pmatrix}
-m_{11} & 0 & \cdots & 0 \\
0 & m_{21} & -m_{22} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & m_{n-1,1} & m_{n-1,2} & \cdots & 0 \\
0 & \cdots & m_{n,1} & m_{n,2} & \cdots & -m_{n,n}
\end{pmatrix},
\begin{pmatrix}
m_{11} & m_{12} & \cdots & m_{1,n-1} & m_{1,n} \\
0 & m_{22} & \cdots & m_{2,n-1} & m_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & m_{n-1,n-1} & m_{n-1,n} \\
0 & \cdots & 0 & m_{n,n}
\end{pmatrix}
\]

\(\epsilon_{ij} \cong (m_{ij}, 0) \ \forall \ i > j, \ \epsilon_{ij} \cong (-m_{ij}, +m_{ij}) \ \forall \ i = j, \ \epsilon_{ij} \cong (0, m_{ij}) \ \forall \ i < j\).

The Lie bracket of \( gl\_n^* \) is given by
Example: The Quantum Grassmannian/1

\( \mathfrak{gl}_n \) is a Lie bialgebra, its dual space \( \mathfrak{gl}_n^* \subset \mathfrak{gl}_n \oplus \mathfrak{gl}_n^* \):

\[
\begin{pmatrix}
-m_{11} & 0 & \cdots & 0 \\
 m_{21} & -m_{22} & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 m_{n-1,1} & m_{n-1,2} & \cdots & 0 \\
 m_{n,1} & m_{n,2} & \cdots & -m_{n,n}
\end{pmatrix}
\begin{pmatrix}
 m_{11} & m_{12} & \cdots & m_{1,n-1} & m_{1,n} \\
 0 & m_{22} & \cdots & m_{2,n-1} & m_{2,n} \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & m_{n-1,n-1} & m_{n-1,n} \\
 0 & 0 & \cdots & 0 & m_{n,n}
\end{pmatrix}
\]

\( \varepsilon_{ij} \cong (m_{ij}, 0) \ \forall \ i > j \), \( \varepsilon_{ij} \cong (-m_{ij}, +m_{ij}) \ \forall \ i = j \), \( \varepsilon_{ij} \cong (0, m_{ij}) \ \forall \ i < j \).

The Lie bracket of \( \mathfrak{gl}_n^* \) is given by

\[
[\varepsilon_{i,j}, \varepsilon_{h,k}] = \delta_{j,h}\varepsilon_{i,k} - \delta_{k,i}\varepsilon_{h,j}, \ \forall \ i \leq j, h \leq k \ \text{and} \ \forall \ i > j, h > k
\]

\[
[\varepsilon_{i,j}, \varepsilon_{h,k}] = \delta_{k,i}\varepsilon_{h,j} - \delta_{j,h}\varepsilon_{i,k}, \ \forall \ i = j, h > k \ \text{and} \ \forall \ i > j, h = k
\]

\[
[\varepsilon_{i,j}, \varepsilon_{h,k}] = 0, \ \forall \ i < j, h > k \ \text{and} \ \forall \ i > j, h < k
\]
$\mathbb{C}_q[G]^\vee$ is generated by:
$\mathbb{C}_q[G]^\vee$ is generated by:

$$\Delta_- := (q-1)^{-1} (D_q^{-1} - 1) , \quad \chi_{ij} := (q-1)^{-1} (x_{ij} - \delta_{ij})$$

$\forall \, i, j = 1, \ldots, n$
$\mathbb{C}_q[G]^{\vee}$ is generated by:

$$\Delta_- := (q - 1)^{-1} (D_q^{-1} - 1) \ , \ \chi_{ij} := (q - 1)^{-1} (x_{ij} - \delta_{ij})$$

where the $x_{ij}$'s are the generators of $\mathbb{C}_q[G]$. 

where
\( \mathbb{C}_q[G]^\vee \) is generated by:

\[
\Delta_- := (q - 1)^{-1} (D_q^{-1} - 1) , \quad \chi_{ij} := (q - 1)^{-1} (x_{ij} - \delta_{ij})
\]

where the \( x_{ij} \)'s are the generators of \( \mathbb{C}_q[G] \).

\( \mathbb{C}_q[G] \) is a quantization of \( G^* \), in fact:
\( \mathbb{C}_q[G] \) is generated by:

\[
\Delta^- := (q - 1)^{-1} (D_q^{-1} - 1) , \quad \chi_{ij} := (q - 1)^{-1} (x_{ij} - \delta_{ij}) \quad \forall \ i, j = 1, \ldots, n
\]

where the \( x_{ij} \)'s are the generators of \( \mathbb{C}_q[G] \).

\( \mathbb{C}_q[G] \) is a quantization of \( G^* \), in fact:

\[
\mathbb{C}_q[G] \uparrow_{q=1} \rightarrow U(\mathfrak{gl}_n^*)
\]

\[
\Delta^- \quad \mapsto \quad -(\epsilon_{1,1} + \cdots + \epsilon_{n,n})
\]

\[
\chi_{i,j} \quad \mapsto \quad \epsilon_{i,j} \quad \forall \ i, j
\]
We give a concrete description of $\mathcal{O}_q(G/P)^\vee$ and prove it is a quantization of $\mathcal{U}(p^\perp)$. 
We give a concrete description of $\mathcal{O}_q(G/P)^\vee$ and prove it is a quantization of $\mathcal{U}(p^\perp)$.

**Proposition**

$$\mathcal{O}_q(G/P)^\vee = \mathbb{k}_q\left\langle \{ \mu_{ij} \}_{i=r+1,\ldots,n} \right\rangle / I_M$$

where $\mu_{ij} := (q - 1)^{-1} t_{ij}$ (for all $i$ and $j$), $I_M$ is the ideal of the Manin relations among the $\mu_{ij}$’s, and $t_{ij} := (-q)^{r-j} d_{ij} d^{-1}$ (for all $i$ and $j$).
We give a concrete description of $\mathcal{O}_q(G/P)^\vee$ and prove it is a quantization of $\mathcal{U}(p^\perp)$.

**Proposition**

$$\mathcal{O}_q(G/P)^\vee = \mathbb{k}_q \left\langle \{ \mu_{ij} \}_{i=r+1}^{r}, \ldots, n \right\rangle / I_M$$

where $\mu_{ij} := (q - 1)^{-1} t_{ij}$ (for all $i$ and $j$), $I_M$ is the ideal of the Manin relations among the $\mu_{ij}$'s, and $t_{ij} := (-q)^{r-j} d_{ij} d^{-1}$ (for all $i$ and $j$).

**Proposition**

$$\mathcal{O}_q(G/P)^\vee \bigg|_{q=1} = \mathcal{U}(p^\perp)$$

as a subalgebra of $\mathbb{C}_q[G]^\vee \bigg|_{q=1} = \mathcal{U}(\mathfrak{gl}_n^*)$, where $p^\perp$ is the orthogonal subspace to $p := \text{Lie}(P)$ inside $\mathfrak{gl}_n^*$. 
Future directions
Future directions

- Generalize the concept of quantum section to supergeometry.
Future directions

- Generalize the concept of quantum section to supergeometry.
  
  **Warning!** Not all homogeneous spaces admit a projective embedding!
Future directions

- Generalize the concept of quantum section to supergeometry. **Warning!** Not all homogeneous spaces admit a projective embedding! Recipe needs to be modified.
Future directions

- Generalize the concept of quantum section to supergeometry.

  **Warning!** Not all homogeneous spaces admit a projective embedding!
  Recipe needs to be modified.

  **Application:** get quantization of conformal space time = flag of $2|0$, $2|1$ spaces into $\mathbb{C}^{4|1}$. 

Future directions

- Generalize the concept of quantum section to supergeometry. **Warning!** Not all homogeneous spaces admit a projective embedding! Recipe needs to be modified.
  
  **Application:** get quantization of conformal space time = flag of 2|0, 2|1 spaces into $\mathbb{C}^{4|1}$.

- Generalize the QDP to quantum supergroups and quantum enveloping superalgebras.
Future directions

- Generalize the concept of quantum section to supergeometry. **Warning!** Not all homogeneous spaces admit a projective embedding! Recipe needs to be modified.
  **Application:** get quantization of conformal space time = flag of $2|0$, $2|1$ spaces into $\mathbb{C}^{4|1}$.
- Generalize the QDP to quantum supergroups and quantum enveloping superalgebras.
- Quantum Principal Bundles and QDP.
Future directions

- Generalize the concept of quantum section to supergeometry.
  Warning! Not all homogeneous spaces admit a projective embedding!
  Recipe needs to be modified.
  Application: get quantization of conformal space time = flag of $2|0$, $2|1$ spaces into $\mathbb{C}^{4|1}$.

- Generalize the QDP to quantum supergroups and quantum enveloping superalgebras.

- Quantum Principal Bundles and QDP.