Vector fields and geodesics in noncommutative geometry

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Related to work with S. Majid on bimodule connections, joint work on quantum mechanics.
four criteria for noncommutative differential geometry

1) There should be a large and diverse collection of examples from many branches of mathematics.

2) Noncommutative geometry should reduce to classical geometry as a special case, though some parts of the theory may become trivial in the classical case.

3) Most constructions in classical differential geometry should have noncommutative geometry analogues.

4) As geometry originated as a practical subject, there should be applications, which historically has meant applications in physics.
Some problems...

1) Algebra maps are not enough!
2) Compact Hausdorff (associative) noncommutative topological spaces seem to be unital $C^*$ algebras - combine this with the DGA approach.
3) Locally compact... ???
4) There are problems with $\ast$-involutions in some examples.
5) There are problems with associativity in some examples. How to generalise?
6) For various problems in discrete geometry and quantum theory $K_0$ is too small...

In this talk we look at (1) and (2).
Let $A$ be a possibly noncommutative algebra. A differential calculus $(\Omega_A, d, \wedge)$ for $A$ is:

1. A graded algebra $\Omega_A = \bigoplus_{n \geq 0} \Omega^n_A$ with $\Omega^0_A = A$.
2. A linear map $d : \Omega^n_A \to \Omega^{n+1}_A$ such that $d^2 = 0$ and
   
   \[ d(\omega_A \wedge \rho) = (d\omega) \wedge \rho + (-1)^n \omega \wedge d\rho, \quad \omega, \rho \in \Omega_A, \ \omega \in \Omega^n_A. \]

3. $A, dA$ generate $\Omega_A$

For a star algebra $A$, a $\star$ differential calculus is one in which

4. the map $a d b \mapsto b^* da^*$ extends to a well defined antilinear map $\Omega^1_A \to \Omega^1_A$. We denote this by $\xi \mapsto \xi^*$.

5. this extends to $\Omega^n_A \to \Omega^n_A$ by

\[ (\xi \wedge \eta)^* = (-1)^{||\xi|| ||\eta||} \eta^* \wedge \xi^*. \]
Given a left $B$-module $E$, a left $B$-covariant derivative is a map $
abla_E : E \rightarrow \Omega^1 \otimes_B E$ which obeys the left Leibniz rule

$$
\nabla_E (b.e) = db \otimes e + b.\nabla_E e
$$

for all $e \in E$ and $b \in B$. Note that in $\Omega^1 \otimes_B E$ the $\otimes_B$ means $\xi. b \otimes e = \xi \otimes b.e$ and corresponds to the fibrewise tensor product.

**Example** Let $B = C^\infty (X)$, the smooth functions on a manifold $X$. Then $\Omega^n_B$ are the usual $n$-forms. A left $B$-module $E$ would be the sections of some bundle on $X$, and given a usual covariant derivative $\nabla_i$ on the bundle for coordinates $x^i$ on $X$, we define

$$
\nabla_E e = dx^i \otimes \nabla_i e
$$
Bimodule covariant derivative

Let $A$ and $B$ be algebras with calculi, and $E$ a $B$-$A$-bimodule, so for $e \in E$ we have $e.a \in E$ for $a \in A$ and $b.e \in E$ for $b \in B$.

For a left covariant derivative $\nabla_E : E \to \Omega^1_B \otimes_B E$ we cannot require both a left and a right Leibniz rule, but we can require a modified right Leibniz rule:

$$\nabla_E(e.a) = \nabla_E(e).a + \sigma_E(e \otimes da), \quad e \in E, \ a \in A$$

where $\sigma : E \otimes_A \Omega^1_A \to \Omega^1_B \otimes_B E$ is a bimodule map. We call $(E, \nabla_E, \sigma_E)$ a bimodule covariant derivative. Given $(F, \nabla_F, \sigma_F)$ on a $A$-$C$-bimodule $F$ we have

$$\nabla_E \otimes F = \nabla_E \otimes \text{id}_F + (\sigma_E \otimes \text{id}_F)(\text{id}_E \otimes \nabla_F)$$

$$\sigma_E \otimes F = (\sigma_E \otimes \text{id})(\text{id} \otimes \sigma_F).$$

Madore, Bresser, Müller-Hoissen, Dimakis, Sitarz, Dubois-Violette, Masson, Mochor, Mourad.
Differentiating module maps

If \((M, \nabla_M, \sigma_M)\) and \((N, \nabla_N, \sigma_N)\) are left \(B-A\)-bimodule connections then given a left module map \(\theta : M \to N\) we define its derivative

\[
\nabla(\theta) = \nabla_N \theta - (\text{id} \otimes \theta) \nabla_M : M \to \Omega^1_B \otimes_B N.
\]

**Proposition** \(\nabla(\theta)\) is a left module map. Further, if \(\theta\) is a bimodule map, then \(\nabla(\theta)\) is a bimodule map if and only if

\[
\sigma_N \circ (\theta \otimes \text{id}) = (\text{id} \otimes \theta) \circ \sigma_M.
\]
In noncommutative geometry the vector fields $\mathcal{X}_A$ are defined as right module maps $\Omega^1_A \to A$, NOT as derivations (Borowiec, Aschieri, Schupp, ...).

We could equally have defined left vector fields. However, we now have a problem defining real vector fields: If $X : \Omega^1_A \to A$ is a right module map then the natural definition of $X^*$ as $X^*(\xi) = X(\xi^*)^*$ would be a left module map.
Classically, given local complex coordinates $z_1, \ldots, z_n$ on a complex manifold we can write a holomorphic vector field as

$$v = f_1(z_1, \ldots, z_n) \frac{\partial}{\partial z_1} + \cdots + f_n(z_1, \ldots, z_n) \frac{\partial}{\partial z_n},$$

where the $f_i(z_1, \ldots, z_n)$ are complex valued holomorphic functions. Differentiating a holomorphic function $a(z_1, \ldots, z_n)$ along a holomorphic vector field gives another holomorphic function, and we would like to have this rather obvious property being true in noncommutative geometry.
Definition
A calculus with an integrable almost complex structure is said to satisfy the factorisability condition if the wedge products
\[ \wedge : \Omega^{0,q} \otimes_A \Omega^{p,0} \to \Omega^{p,q}, \quad \wedge : \Omega^{p,0} \otimes_A \Omega^{0,q} \to \Omega^{p,q} \]
are bimodule isomorphisms for all \( p, q \geq 0 \). Denote the inverses
\[ \Theta^{0qp0} : \Omega^{p,q} \to \Omega^{0,q} \otimes_A \Omega^{p,0}, \quad \Theta^{p00q} : \Omega^{p,q} \to \Omega^{p,0} \otimes_A \Omega^{0,q} \]

Example
To get the map \( \Theta^{0qp0} \) on a classical complex manifold with local complex coordinates \( z_i \), just permute all the \( d\bar{z}_i \) to the left, with the appropriate power of -1, and replace the \( \wedge \) separating the \( d\bar{z}_i \) from the \( dz_i \) by \( \otimes \). For example,
\[
dz_1 \wedge \dbar_2 \longleftrightarrow -d\bar{z}_2 \otimes dz_1, \\
d\bar{z}_1 \wedge dz_1 \wedge d\bar{z}_2 \longleftrightarrow -d\bar{z}_1 \wedge d\bar{z}_2 \otimes dz_1.
\]
We require that a holomorphic vector field $\nu \in \mathfrak{X}^{1,0}$ satisfy $
abla \text{ev}(\nu \otimes da) = 0$ for all $a \in A$ with $\nabla a = 0$. Note that $\text{ev}(\nu \otimes da) = \text{ev}(\nu \otimes \partial a)$ here since $\nu \in \mathfrak{X}^{1,0}$. However there may not be any global non-constant holomorphic functions, and we do not have the luxury of working locally, so we need to have a condition which is more general. We use a right $\nabla$-operator $\diamondsuit : \Omega^{1,0} \to \Omega^{1,0} \otimes_A \Omega^{0,1}$ making $\Omega^{1,0}$ into a holomorphic right $A$-module, with dual left $\nabla$-operator $\dagger : \mathfrak{X}^{1,0} \to \Omega^{0,1} \otimes_A \mathfrak{X}^{1,0}$. Here

$$
\nabla \text{ev}(\nu \otimes \partial a) = (\text{id} \otimes \text{ev})(\diamondsuit \nu \otimes \partial a) + (\text{ev} \otimes \text{id})(\nu \otimes \dagger \partial a).
$$

We see that if $\dagger \partial a = 0$ for all $a \in A$ with $\nabla a = 0$ then $\dagger \nu = 0$ implies that $\text{ev}(\nu \otimes \partial a)$ is holomorphic. Thus we have a holomorphic vector field being given by a left $\nabla$-operator with an extra condition attached. Fortunately when the factorisation condition holds we have a canonical choice for $\dagger$. 
Lemma

Suppose the calculus is factorisable. Then
\[ \clubsuit : \Omega^{p,0} \to \Omega^{p,0} \otimes_A \Omega^{0,1} \]
defined by \( \clubsuit = (-1)^p \Theta^{p001} \overline{\partial} \) and
\[ \tilde{\sigma} = (-1)^p \Theta^{p001} \circ \wedge \]
makes \( \Omega^{p,0} \) into a right holomorphic A-bimodule.

Using the consequence \( \partial \overline{\partial} + \overline{\partial} \partial = 0 : A \to \Omega^2 \) we also have
\[
\clubsuit (\partial a) = -\Theta^{1001} (\overline{\partial} \partial a) = \Theta^{1001} (\partial \overline{\partial} a) = 0
\]
in the case where \( \overline{\partial} a = 0 \), so \( \clubsuit \) on \( \mathcal{X}^{1,0} \) gives a satisfactory
definition of holomorphic vector fields.
A classical bimodule covariant derivative

Classically we have algebras smooth real valued functions on some manifold. Then a classical $A$-$A$-bimodule covariant derivative is normally really boring as we get $e.a = a.e$ and then $\sigma_E(e \otimes da) = da \otimes e$. But we are going to look at the case where the right and left algebras are not the same...
A classical bimodule covariant derivative

Give $\mathbb{R}^n$ coordinates $(x^1, \ldots, x^n)$ and $\mathbb{R}$ a coordinate $t$, and consider a path $\gamma = (\gamma^1, \ldots, \gamma^n) : \mathbb{R} \to \mathbb{R}^n$. Define $\tilde{\gamma} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R})$ by $\tilde{\gamma}(a) = a \circ \gamma$, so $\tilde{\gamma}(x^i) = \gamma^i$. Now $M = C^\infty(\mathbb{R})\tilde{\gamma}$ is a $C^\infty(\mathbb{R})$-$C^\infty(\mathbb{R}^n)$ bimodule, which is $C^\infty(\mathbb{R})$ as a vector space with product left action by $C^\infty(\mathbb{R})$. The right action is $f \triangleleft a = f \tilde{\gamma}(a)$, so $f \triangleleft x^i = f \gamma^i$.

Give $M$ a bimodule connection by

$$\nabla_M(f) = \dot{f}(t) \, dt \otimes 1$$

$$\sigma_M(1 \otimes dx^i) = \nabla_M(1 \triangleleft x^i) - \nabla_M(1) \triangleleft x^i = \nabla_M(\gamma^i) = \dot{\gamma}^i(t) \, dt \otimes 1$$

using dot for time derivative. The velocity $V$ of the path is

$$V(dx^i) = (\text{ev} \otimes \text{id})(\frac{\partial}{\partial t} \otimes \sigma_M(1 \otimes dx^i)) = \dot{\gamma}^i(t)$$
Rate of change along a curve

**Proposition** Given a vector bundle on $\mathbb{R}^n$ with sections $E$ and connection $\nabla_E$ we write $\nabla_E(e) = dx^i \otimes \nabla_{E_i}(e)$. Then the rate of change of $e \in E$ along $\gamma$ is

$$
\left( \text{ev} \otimes \text{id} \right) \left( \frac{\partial}{\partial t} \otimes \nabla_M \otimes E \left( 1 \otimes e \right) \right) = \dot{\gamma}^i(t) \otimes \nabla_{E_i}(e).
$$

Using the vector fields $\mathfrak{X}(\mathbb{R}^n)$ with a connection given by the usual Christoffel symbols, for $V$ as before

**Corollary**

$$
\nabla_M \otimes \mathfrak{X}(\mathbb{R}^n)(V) = \left( \frac{d^2 \gamma^i(t)}{dt^2} + (\Gamma^i_{jk} \circ \gamma) \frac{d\gamma^j(t)}{dt} \frac{d\gamma^k(t)}{dt} \right) dt \otimes 1 \otimes \frac{\partial}{\partial x^i}.
$$

Thus $\nabla_M \otimes \mathfrak{X}(\mathbb{R}^n)(V) = 0$ is just the usual equation of a geodesic.
A short statement

There is an alternative form which is simpler to interpret in the language of bimodule connections, and can be seen to be equivalent on using connections on dual bundles.

**Proposition** The equation $\nabla(\sigma_M) = 0$ is equivalent to the parallel transport of the velocity vector.
In quantum mechanics, the Heisenberg uncertainty principle means that the idea of a geodesic as a single path has to be replaced by a more uncertain or ‘probabilistic’ idea, as we cannot precisely measure both position and velocity (momentum) at the same time, and the path depends on both.

Previously we constructed $M$ from a path $\gamma : \mathbb{R} \to \mathbb{R}^n$ which induced an algebra map $\tilde{\gamma} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R})$. In noncommutative geometry it will prove impractical to follow this pattern, as there are simply not enough such algebra maps – instead we look for completely positive maps.
The KSGNS construction for completely positive maps

**Definition** For $A$ and $B$ dense $\ast$-subalgebras of some $C^*$ algebras and a $B$-$A$-bimodule $M$, a positive semi-inner product on $M$ is a bimodule map $\langle , \rangle : M \otimes_A \overline{M} \to B$ with $\langle m, m' \rangle^* = \langle m', m \rangle$ and $\langle m, m \rangle$ positive in $B$ for all $m, m' \in M$.

The KSGNS construction says that completely positive maps $\psi : A \to B$ for $C^*$-algebras $A$ and $B$ are all given by $B$-$A$-bimodules $M$ with positive inner products using the formula $\psi(a) = \langle m.a, m \rangle$ for some $m \in M$.

**Example** For the previous example $M = C^\infty(\mathbb{R})$ we have $\langle , \rangle : M \otimes_{C^\infty(\mathbb{R}^n)} \overline{M} \to C^\infty(\mathbb{R})$ given by $\langle f, g \rangle = f g^*$. 
A modification of a representation by Woronowicz

(F. Alghamdi, Ph.D. thesis)

Let $E$ be a complex vector space with basis $\psi_{n,k}$ for integers $k \in \mathbb{Z}$ and $n \geq 0$. Then $E$ has a right action of the Hopf algebra $A = \mathbb{C}[z, z^{-1}]$ given by $\psi_{n,k} \triangleright z^j = \psi_{n,k+j}$. Define a $\mathbb{C}[z, z^{-1}]$-valued inner product $\langle , \rangle : \overline{E} \otimes E \to \mathbb{C}[z, z^{-1}]$ making $E$ into an inner product $\mathbb{C}[z, z^{-1}]$-module by

$$\langle \psi_{m,s}, \psi_{n,k} \rangle = z^{k-s} \delta_{n,m}.$$

For $q \in \mathbb{R}$ with $|q| > 1$, there is a left action of $B = \mathbb{C}_q[SU_2]$ on $E$ by

$$a \triangleright \psi_{n,k} = \sqrt{1 - q^{-2n}} \psi_{n-1,k}, \quad c \triangleright \psi_{n,k} = q^{-n} \psi_{n,k+1}$$

$$a^* \triangleright \psi_{n,k} = \sqrt{1 - q^{-2n-2}} \psi_{n+1,k}, \quad c^* \triangleright \psi_{n,k} = q^{-n} \psi_{n,k-1}.$$
Now examine the conditions to have a bimodule covariant derivative

\[ \tilde{\sigma}_E : \Omega^1_{\mathbb{C}_q[SU_2]} \mathbb{C}_q[SU_2] \otimes E \rightarrow E \otimes \Omega^1_{\mathbb{C}[z,z^{-1}]} \mathbb{C}_q[S^1]. \]

Now construct a completely positive map extending to a cochain map on differential forms. Set \( e_n = \psi_{n,n} \) for some \( n \geq 0 \). Then \( \tilde{\nabla}_E(e_n) = 0 \) and \( \phi_n \) is defined by

\[ \phi_n(y) = \langle \overline{e_n}, y \triangleright e_n \rangle = \langle \overline{\psi_{n,n}}, y \triangleright \psi_{n,n} \rangle. \]

For the standard basis for \( \mathbb{C}_q[SU_2] \), we have \( \phi_n(a^m b^r c^s) = 0 \) and \( \phi_n(d^m b^r c^s) = 0 \) for \( m > 0 \) and

\[ \phi_n(b^r c^s) = q^{-n(r+s)} (-q^{-1})^r z^{s-r}. \]

This is not an algebra map. The value of \( \phi_n \) on 1-forms is given by \( \phi_n \) vanishing on any multiple of \( e^\pm \). For \( e^0 \), we calculate

\[ \phi_n(y . e^0) = (\langle , \rangle \otimes \text{id})(\overline{\psi_{n,n}} \otimes \tilde{\sigma}_E(y . e^0 \otimes \psi_{n,n})) = \phi_n(y) . \omega^0 = \phi_n(y) . z^{-1} \, dz \]

as the degree 1 part of a cochain map

\[ \Omega(\mathbb{C}_q[SU_2]) \rightarrow \Omega(\mathbb{C}_q[S^1]). \]
Another bimodule $M$

For a unital algebra $A$ with calculus $\Omega_A$ we set $M = C^\infty(\mathbb{R}) \otimes A$ regarded as a $C^\infty(\mathbb{R})$-$A$-bimodule. Then a general left bimodule connection on $M$ is, for $c \in C^\infty(\mathbb{R}) \otimes A$ and $\xi \in \Omega^1_A$

$$\nabla_M(c) = dt \otimes (bc + \frac{\partial c}{\partial t} + K(dc)) , \quad \sigma_M(1 \otimes \xi) = dt \otimes K(\xi)$$

for some $b \in C^\infty(\mathbb{R}) \otimes A$ and $K \in C^\infty(\mathbb{R}) \otimes \mathcal{X}_A$. [We will usually say $b$ and $K$ are time dependent.]

**Proposition** $\nabla (\sigma_M) = 0$ if and only if both

$$K(K \otimes \text{id})\sigma_{\Omega^1_A} = K(K \otimes \text{id})$$

$$(\frac{\partial K(\xi)}{\partial t}) = K(b\xi) - bK(\xi) + K(K \otimes \text{id})\sigma_{\Omega^1_A}^{-1} \nabla_{\Omega^1_A} (\xi) - K(dK(\xi)) . (1)$$

Further, if $K$ satisfies (1) for $\xi \in \Omega^1_A$ then it also satisfies it for $\xi a \in \Omega^1_A$ for all $a \in A$, so it is only necessary to verify (1) for a collection of right generators of $\Omega^1_A$. 
The new $M$ for the classical $A = C^\infty(\mathbb{R}^n)$ case

Equation (1) for the time dependent vector field $K$ becomes

$$\frac{\partial K^i}{\partial t} + K^s K^i,_s + K^k K^j \Gamma^i_{jk} = 0 \quad (2)$$

Suppose we start a point at $x(0) \in \mathbb{R}^n$ for $t = 0$ and move it according to the vector field $\frac{dx}{dt} = K(x)$. As the point moves, the ‘convective derivative’ from fluid mechanics gives

$$\frac{dK^i(x)}{dt} = \frac{\partial K^i}{\partial t} + K^s K^i,_s$$

and so (2) is

$$\frac{dK^i(x)}{dt} + K^k K^j \Gamma^i_{jk} = 0$$

the usual equation for the velocity being parallel transported.

We get the velocity field for particles obeying geodesic motion starting at arbitrary points.
The time dependence of the positive map

Given $\nabla (\sigma_M) = 0$ the time evolution of $\phi(a) = \langle m.a, \overline{m} \rangle$ is given by $\nabla_M(m) = 0$. For example for a positive function on $C^\infty(\mathbb{R}^3)$

$$\phi(t)(f) = \int_{\mathbb{R}^3} |m(t)|^2 f \, d^3 x$$

We use the connection $\nabla_M(m) = m b + K(dm) + \frac{\partial m}{\partial t}$. Finally the inner product above is preserved by the connection if

$$b + b^* = \frac{\partial K^i}{\partial x^i}$$

and this shows that $\phi$ is normalised, i.e. if $\phi(1) = 1$ at time 0 it remains true - we get an evolution on states.

In general $\nabla_M$ preserves the inner product if for all $a \in A$ and $\xi \in \Omega^1_A$ (the last bit giving a reality condition for the vector field)

$$\langle (ba + K(da) + ab^*), \overline{1} \rangle = 0 = \langle K(\xi^*), \overline{1} \rangle .$$
The $C(\mathbb{Z}_n)$ example

Take the finite group $(\mathbb{Z}_n, +)$ and the algebra $A = \mathbb{C}(\mathbb{Z}_n)$ of functions $f : \mathbb{Z}_n \to \mathbb{C}$ with basis $\delta_i$ for $0 \leq i \leq n - 1$, which is the function $\delta_i(j) = \delta_{i,j}$. This has a calculus where $\Omega^1_A$ has two non-central generators $e_{+1}$ and $e_{-1}$, where

$$e_a \cdot f = R_a(f) e_a, \quad df = e_{+1}(f - R_{-1}(f)) + e_{-1}(f - R_{+1}(f)),$$

and $R_a(f)(i) = f(i + a)$ (in mod $n$ arithmetic). (This is a Hopf algebra with bicovariant calculus.) Take $\kappa_{+1}$ and $\kappa_{-1}$ to be the dual basis of vector fields to $e_{+1}$ and $e_{-1}$. The $*$-operation is $e_{\pm 1}^* = -e_{\mp 1}$.
Equations for the $C(\mathbb{Z}_n)$ example

Set $A = \mathbb{C}(\mathbb{Z}_n)$ and $M = C^\infty(\mathbb{R}) \otimes \mathbb{C}(\mathbb{Z}_n)$ with inner product

$$\langle a, \overline{c} \rangle = \sum a(i) c(i)^*$$

For the vector field $K$ set $K_\pm = K(e_\pm)$, and we get $K_- = -R_1(K_+^*)$ as the reality condition for $K$ relative to the inner product and the divergence condition

$$0 = b + b^* + K_+ - R_{+1}(K_+) + K_- - R_{-1}(K_-).$$

The equations for the vector field are

$$\frac{\partial K_+}{\partial t} = K_+ \left( R_{-1}(b + K_+ + K_-) - (b + K_+ + K_-) \right)$$

$$\frac{\partial K_-}{\partial t} = K_- \left( R_{+1}(b + K_+ + K_-) - (b + K_+ + K_-) \right)$$

(3)

This gives coupled nonlinear ODEs for $2n$ functions $K_\pm(i)$. 
Numerical solution for $C(\mathbb{Z}_3)$

For the case $n = 3$ we solve this numerically with the initial conditions for the vector field $K$ and $m \in M$ at $t = 0$

$$K_-(1) = 1, \quad K_-(2) = e^{2i}, \quad K_-(0) = e^{3i}, \quad K_+(1) = -e^{-3i}, \quad K_+(2) = -1,$$

$$K_+(0) = -e^{-2i}, \quad m(0) = \frac{1}{\sqrt{2}}, \quad m(1) = 0, \quad m(2) = \frac{1}{\sqrt{2}}.$$

The graph shows $|m(0)(t)|^2$, $|m(0)(t)|^2 + |m(1)(t)|^2$ and $|m(0)(t)|^2 + |m(1)(t)|^2 + |m(2)(t)|^2$ for $t \in [0, 10]$, i.e. the time variation of the state.
An $M_2(\mathbb{C})$ example

Set $A = M_2(\mathbb{C})$, the 2 by 2 complex matrices. This is given a calculus where $\Omega^1_A$ is freely generated by two central generators $s^1$ and $s^2$, with

$$da = s^1 [E_{12}, a] + s^2 [E_{21}, a]$$

where $E_{ij} \in M_2(\mathbb{C})$ has zero entries except for 1 in the $ij$ position. We take $e_1, e_2$ to be the (central) dual basis of vector fields to $s^1, s^2$. The $\ast$-operation is $s^1 \ast = -s^2$.

Yet another bimodule: Use a different construction containing less information; set $N = \text{Row}^2(\mathbb{C})$ (the two dimensional row vectors), and then the inner product $\langle n', \bar{n} \rangle = n' n^\ast \in \mathbb{C}$ gives the pure states on $M_2(\mathbb{C})$ as $\phi(r) = \langle nr, \bar{n} \rangle / \langle n, \bar{n} \rangle$. The space of pure states is given by $[n] \in \mathbb{C}P^1$. 
Yet another bimodule

Set $M = C^\infty(\mathbb{R}) \otimes N$, and then the possible bimodule maps

$\sigma_M : M \otimes A \Omega^1_A \rightarrow \Omega^1(\mathbb{R}) \otimes C^\infty(\mathbb{R}) M$ are given by

$$\sigma_M(w \otimes s_i) = dt \otimes Q_i w$$

for $w \in \text{Row}^2(\mathbb{C})$ and $Q_i \in C^\infty(\mathbb{R})$. Then for $f \in C^\infty(\mathbb{R})$

$$\nabla_M(fw) = \frac{df}{dt} dt \otimes w - dt f \otimes w( Q_1 E_{12} + Q_2 E_{21} + Q_0 I_2)$$

for some $Q_0 \in C^\infty(\mathbb{R})$. On $\Omega^1_A$ use the connection $\nabla_A(s^i) = 0$. On checking a braid relation, to show that $\mathcal{W}(\sigma_M)$ vanishes it is only necessary to show it vanishes on generators, thus $Q_1$ and $Q_2$ are constant and $Q_0$ is arbitrary. The dynamics of $m = (z(t), 1)$ is by Möbius transformations

$$z(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright z(0) = \frac{az(0) + b}{cz(0) + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \exp\left( t \begin{pmatrix} 0 & Q_2 \\ Q_1 & 0 \end{pmatrix} \right)$$
Let $M$ be a symplectic manifold with coordinates $x^\mu$, $\mu = 1, \ldots, 2n$, symplectic form $\omega_{\mu\nu}$ and its inverse, the Poisson tensor $\omega^{\mu\nu}$, and let $\nabla$ be a symplectic connection (i.e. torsion free and preserving the symplectic form) with Christoffel symbols $\Gamma^\mu_{\nu\rho}$. A function $h \in C^\infty(M)$ gives rise to a vector field $X_h$ by $X_h^\mu = \omega^{\mu\nu} h_\nu$, the Poisson tensor applied to $dh$ (we use $h_j$ for the partial derivative of $h$ with respect to $x^j$). If $h$ is any function then its Hamiltonian vector field $X_h$ is not autoparallel as

$$\nabla_{X_h} X_h^\mu = \omega^{\alpha\beta} h_\beta \omega_{\mu\nu} \nabla_\alpha h_\nu = \omega^{\alpha\beta} h_\beta \omega_{\mu\nu} (h_\nu \gamma - \Gamma^\gamma_{\beta\nu} h_\gamma) = -g^{\mu\nu} h_\nu$$

where $\nabla_\mu = \nabla \frac{\partial}{\partial x^\mu}$ we define

$$(dx^\mu, dx^\nu) = g^{\mu\nu} = \omega^{\mu\gamma} \omega_{\nu\rho} \nabla_\gamma h_\rho$$

as the possibly degenerate metric inner product, which is symmetric as $\nabla$ is torsion free.
Next, this obstruction to $X_h$ being autoparallel can be resolved by adding an extra dimension $\theta'$ to the cotangent bundle. This is taken to commute with functions and $\theta' \wedge \theta' = d\theta' = 0$ in the exterior algebra, and has a dual central vector field $\kappa$.

$$\tilde{\nabla} dx^\nu = dx^\mu \otimes \nabla_\mu dx^\nu - \Gamma^\nu_{0\alpha} \theta' \otimes dx^\alpha - \Gamma^\nu_{\alpha 0} dx^\alpha \otimes \theta'$$\hspace{1cm} \nabla \theta' = 0.

Thus, in terms of Christoffel symbols, we have introduced a label 0 to stand for the hypothetical new ‘coordinate’ direction, with $\Gamma^0_{**} = 0$. We similarly define an extended vector field

$$\tilde{X}_h = X_h + \kappa.$$

**Lemma**

For generic $h$, $\tilde{X}_h$ is autoparallel with respect to $\tilde{\nabla}$ if and only if $\Gamma^\mu_{\alpha 0} + \Gamma^\mu_{0\alpha} = g^{\mu\beta} \omega_{\beta\alpha}$. 
Schrödinger equation

\[
[x^i, p_j] = i\hbar \delta^i_j, \quad [x^i, x^j] = [p_i, p_j] = 0
\]

\[
h = \frac{p_1^2 + \ldots + p_n^2}{2m} + V(x^1, \ldots, x^n)
\]

Now we choose \( M = L^2(\mathbb{R}^n) \otimes C^\infty(\mathbb{R}) \) to be the standard Schrödinger representation. Following the geodesic interpretation, we regard \( M \) as a right \( C^\infty(\mathbb{R}) \) module by a trivial product, and the equation of motion for \( \Psi \) is given by a right connection \( \tilde{\nabla} : M \to M \otimes C^\infty(\mathbb{R}) \Omega^1_\mathbb{R} \) by \( \tilde{\nabla}_M(\Psi) = 0 \). (Here \( \Omega^1_\mathbb{R} \) denotes the usual 1-forms on \( \mathbb{R} \).) We can encode the Schrödinger equation as \( \tilde{\nabla}(\psi) = 0 \) for the following covariant derivative

\[
\tilde{\nabla}(\psi) = \frac{\partial}{\partial t} \psi - \frac{1}{i\hbar} h \psi.
\]
Proposition

There is a unique differential calculus on the Heisenberg algebra such that $\tilde{\nabla}_M$ is an $A-C^\infty(\mathbb{R})$-bimodule connection with $\tilde{\sigma}_M(\theta' \otimes \cdot) = \text{id}$ and $\tilde{\sigma}_M$ is injective, namely

$$[dp_i, p_j] = -i\hbar \frac{\partial^2 V}{\partial x^i \partial x^j} \theta', \quad [dp_i, x^j] = [dx^i, p_j] = 0, \quad [dx^i, x^j] = -\frac{i\hbar}{m} \delta_{ij} \theta'.$$

This is inner with $\theta = \frac{1}{\hbar}(x^i dp_i - p_i dx^i)$.

To solve the quantum autoparallel equation in a form that deforms the classical connection, we are naturally led to set

$$\tilde{\nabla}_A(dx^i) = \frac{1}{m} \theta' \otimes dp_i, \quad \tilde{\nabla}_A(dp_i) = -\frac{\partial^2 V}{\partial x^i \partial x^j} \theta' \otimes dx^j + \frac{i\hbar}{2m} \frac{\partial^3 V}{\partial x^i \partial x^i \partial x^j} \theta' \otimes \theta'$$
Proposition

Under our assumptions there is a unique bimodule map $\Omega^1_A \to A$ (a left and right quantum vector field)

$$\tilde{X}_h(\theta') = 1, \quad \tilde{X}_h(dp_i) = -\frac{\partial V}{\partial x^i}, \quad \tilde{X}_h(dx^i) = \frac{1}{m} p_i$$

such that $\tilde{\sigma}_M(\xi \otimes \psi) = \tilde{X}_h(\xi)\psi$ for all $\xi \in \Omega^1_A$. In this case $\tilde{\nabla}(\tilde{\sigma}_M) = 0$ is equivalent to the ‘quantum autoparallel equation’

$$\left(\tilde{X}_h \otimes \tilde{X}_h\right)\tilde{\nabla}_A\xi = \frac{1}{i\hbar} [\tilde{X}_h(\xi), h]$$

or explicitly

$$\left(\tilde{X}_h \otimes \tilde{X}_h\right)\tilde{\nabla}_A(\theta') = 0,$$

$$\left(\tilde{X}_h \otimes \tilde{X}_h\right)\tilde{\nabla}_A(dp_i) = -\frac{1}{m} \frac{\partial V}{\partial x^i \partial x^j} p_j + \frac{i\hbar}{2m} \frac{\partial^3 V}{\partial x^i \partial x^j \partial x^l},$$

$$\left(\tilde{X}_h \otimes \tilde{X}_h\right)\tilde{\nabla}_A(dx^i) = -\frac{1}{m} \frac{\partial V}{\partial x^i}.$$
Klein-Gordon???
There are paths and connections - is there holonomy?

The *time* algebra $B$ could be different from $C^\infty(\mathbb{R})$, possibly Hopf algebras with $\Omega_B^1$ singly generated??

Look at the deformed cases to see what the physics is. Could varying normalisation be particle creation or destruction?

Quantum gravity is not necessarily *Quantum gravity*. Current experiments on entangled states are approaching the scale where quantised gravitational fields may become measurable. Independently of the Planck length, Quantum gravity may be coming soon.

The importance of position in quantum field theory...