Quantized pseudo-fixed-point subalgebras
Towards a classification of quasitriangular coideal subalgebras

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1 Introduction and review

2 Pseudo-involutions and compatible decorations

3 Pseudo-fixed-point subalgebras and generalized Satake diagrams

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4 Quantized pseudo-fixed-point subalgebras
A Dynkin diagram can be used to define several objects.

1. a Kac-Moody Lie algebra $\mathfrak{g}$ (Killing, Cartan; Kac, Moody);

2. its quantization $U_q\mathfrak{g}$, a non-cocommutative bialgebra (Drinfeld, Jimbo, ...);

3. its quasitriangular structure, nl. universal R-matrix (Drinfeld, Lusztig, ...) - a solution $\otimes$ of the Yang-Baxter equation

4. In finite and affine type: factorization of the quasi R-matrix (Levendorskii & Soibelman, Kirillov & Reshetikhin; Khoroshkin & Tolstoy).
Similarly, a Satake diagram (a decorated Dynkin diagram) gives rise to the following objects.

1. An involutive automorphism $\theta$ of $\mathfrak{g}$ “of the second kind” [Araki; Kac & Wang, Kolb] and hence the corresponding fixed-point subalgebra $\mathfrak{k} = \mathfrak{g}^\theta$; this is in fact a classification.

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2. The quantization of $\mathfrak{k}$: the coideal subalgebra $B = U_q(\mathfrak{g}^\theta)$ [Letzter; Kolb]. It has the property $B \cap U_q \mathfrak{h} = U_q(\mathfrak{h}^\theta)$.
   Cf. [Noumi, Sugitani & Dijkhuizen, 199X].
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3. In finite type, a quasitriangular structure: the universal K-matrix [Balagović & Kolb] - a solution of the reflection equation

\[
\begin{pmatrix}
\begin{array}
| & | \\
| & |
\end{array}
\end{pmatrix}
= 
\begin{pmatrix}
\begin{array}
| & | \\
| & |
\end{array}
\end{pmatrix}
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Similarly, a Satake diagram (a decorated Dynkin diagram) gives rise to the following objects.

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\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{reflection_equation.png}
\end{array}
\end{align*}
\]

4. In finite type: factorization of the quasi K-matrix [Dobson & Kolb].
Natural questions:

- Are there more coideal subalgebras $B$ of $U_q\mathfrak{g}$ with the property $B \cap U_q\mathfrak{h} = U_q(\mathfrak{h}^\theta)$ and, at least for finite type, possessing a universal $K$-matrix? Can we define these new $B$ using more general combinatorial datum? Can we classify all such $B$ (old and new)?

Joint work with Vidas Regelskis (University of Hertfordshire):


V. Regelskis and B. Vlaar, “Quantized pseudo-fixed point subalgebras”, in progress.

And what about a universal $K$-matrix for infinite type, in particular affine? Joint work with Andrea Appel (University of Edinburgh), in progress.
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- What else does this more general combinatorial datum define/classify? In particular, what about the corresponding Lie subalgebra of $\mathfrak{g}$?

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Kac-Moody algebras: basic Lie-theoretic notation/facts

I Set of nodes of a Dynkin diagram.
A Symmetrizable, indecomposable generalized Cartan matrix with entries $a_{ij} \in \mathbb{Z}$ ($i, j \in I$). Note: we assume $a_{ii} = 2$.
$g$ Corresponding Kac-Moody Lie algebra/$\mathbb{C}$.
$h$ Its standard Cartan subalgebra of dimension $|I| + \operatorname{cork}(A)$.
$e_i, f_i$ ($i \in I$) Chevalley-Serre generators of $g'$. $h_i := [e_i, f_i] \in h$.
$n^\pm$ Standard upper/lower nilpotent subalgebras:
$n^+ = \langle \{e_i\}_{i \in I} \rangle$ and $n^- = \langle \{f_i\}_{i \in I} \rangle$.
$W := \langle \{r_i\}_{i \in I} \rangle$ corresponding Weyl group, acting on $h$ and $h^*$.
$\Phi \subset h^*$. Root system with simple roots $\{\alpha_i\}_{i \in I}$. Preserved by $W$.

Triangular decomposition:

$$g = n^+ \oplus h \oplus n^-.$$ 

Root space decomposition:

$$g = h \oplus \bigoplus_{\alpha \in \Phi} g_\alpha.$$
Lie algebra automorphisms of $\mathfrak{g}$ and $\mathfrak{g}'$

**First kind**

Call $\theta \in \text{Aut}(\mathfrak{g}(^{(i)}))$ of the first kind if $\dim(\theta(n^+) \cap n^-) < \infty$.

Example: any diagram automorphism $\tau$ acting via $\tau(e_j) = e_{\tau(j)}$ etc.

Warning: if $\det(A) = 0$ this is not automatically an automorphism of $\mathfrak{g}$.

[Kac & Wang, 1992]
Lie algebra automorphisms of $\mathfrak{g}$ and $\mathfrak{g}'$

First kind

Call $\theta \in \text{Aut}(\mathfrak{g}^{(\tau)})$ of the first kind if $\dim(\theta(\mathfrak{n}^+) \cap \mathfrak{n}^-) < \infty$.

Example: any diagram automorphism $\tau$ acting via $\tau(e_j) = e_{\tau(j)}$ etc.

Warning: if $\det(A) = 0$ this is not automatically an automorphism of $\mathfrak{g}$.

Second kind

Call $\theta \in \text{Aut}(\mathfrak{g}^{(\tau)})$ of the second kind if $\dim(\theta(\mathfrak{n}^+) \cap \mathfrak{n}^+) < \infty$.

Example: Chevalley involution $\omega$ defined by

$$
\omega(e_j) = -f_j, \quad \omega(f_j) = -e_j, \quad \omega(h) = -h \quad (h \in \mathfrak{h})
$$

[Kac & Wang, 1992]
**Lie algebra automorphisms of $g$ and $g'$**

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**Second kind**

Call $\theta \in \text{Aut}(g^{(i)})$ of the second kind if $\dim(\theta(n^+) \cap n^+) < \infty$.

Example: Chevalley involution $\omega$ defined by

$$\omega(e_j) = -f_j, \quad \omega(f_j) = -e_j, \quad \omega(h) = -h \quad (h \in h)$$

- If $\dim(g) < \infty$ these two types coincide.
- If $\dim(g) = \infty$ these two types partition $\text{Aut}(g)$.
- In either case: \{automs. of 2nd kind\} $= \omega \circ \{\text{automs. of 1st kind}\}$.

[Kac & Wang, 1992]
Pseudo-involutions

Any semisimple automorphism of $\mathfrak{g}$ of the 2nd kind, up to conjugacy, has a combinatorial factorization [Kac & Wang, 1992 (4.38 & 4.39)]. This includes involutive automorphisms of $\mathfrak{g}$ of the 2nd kind for which [Kolb, 2014] gives a precise classification statement in terms of Satake diagrams.
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**Definition**

Call $\sigma \in \text{Aut}(\mathfrak{g})$ a **pseudo-involution** if it is of the 2nd kind, semisimple and $\exists$ Cartan $\mathfrak{t} \subset \mathfrak{g}$ such that $\sigma(\mathfrak{t}) = \mathfrak{t}$ and $\sigma^2|_{\mathfrak{t}} = \text{id}$. 
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**Definition**

Call $\sigma \in \text{Aut}(\mathfrak{g})$ a **pseudo-involution** if it is of the 2nd kind, semisimple and $\exists$ Cartan $t \subset \mathfrak{g}$ such that $\sigma(t) = t$ and $\sigma^2|_t = \text{id}$.

For pseudo-involutions we have $t = t^\sigma \oplus t^{-\sigma}$. Let us build them from combinatorial datum.
Compatible decorations

For \( X \subseteq I \) of finite type: \( g_X := \langle \{e_i, f_i\}_{i \in X} \rangle \subseteq g \) is fin.-dim. semisimple
\( W_X := \langle \{r_i\}_{i \in X} \rangle \subseteq W \) is finite.

- Unique longest element \( w_X \in W_X \) (involution).
- \( \exists \) diagram automorphism \( \text{op}_X : X \to X \) s.t. \( w_X(h_i) = -h_{\text{op}_X(i)} \).
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- $\exists$ diagram automorphism $\text{op}_X : X \to X$ s.t. $w_X(h_i) = -h_{\text{op}_X(i)}$.

Consider “compatible decorations” of Dynkin diagrams:

$$CDec(A) = \{(X, \tau) \mid \tau : I \to I \text{ diagram automorphism, } \tau^2 = \text{id, } X \subseteq I \text{ of finite type, } \tau(X) = X, \tau|_X = \text{op}_X\}.$$ 

- Satake diagrams are examples of compatible decorations.
- Elements of $X \sim$ filled nodes. Nontrivial $\tau$-orbits $\sim$ arrows.

To build a pseudo-involution and to control its action on $g_X$, we would like to lift the action of $w_X$ on $\mathfrak{h}$ to an action on $g$. 

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Quantized pseudo-fixed point subalgebras
For $i \in I$ let $m_i := \exp(e_i) \exp(-f_i) \exp(e_i) \in G$ so that
\[
\text{Ad}(m_i) = \exp(\text{ad}(e_i)) \circ \exp(\text{ad}(-f_i)) \circ \exp(\text{ad}(e_i)) \in \text{Aut}(g)
\]
- $\text{Ad}(m_i)|_{\mathfrak{h}} = r_i$ ($W$ acting on $\mathfrak{h}$).
- $\text{Ad}(m_i)(\mathfrak{g}_\alpha) = \mathfrak{g}_{ri(\alpha)}$ ($W$ acting on $\mathfrak{h}^*$) ...so $\text{Ad}(m_i)$ is of 1st kind.
- $\{\text{Ad}(m_i)\}_{i \in I}$ satisfy same braid relations as $\{r_i\}_{i \in I}$.
- $\text{Ad}(m_i^2)|_{\mathfrak{g}_\alpha} = (-1)^{\alpha(h_i)}$.

[Kac & Wang, 1992]
[Back-Valente, Bardy-Panse, Ben Massaoud & Rousseau, 1995]
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\]

- \( \text{Ad}(m_i)|_{\mathfrak{h}} = r_i \) (\( W \) acting on \( \mathfrak{h} \)).
- \( \text{Ad}(m_i)(g_\alpha) = g_{r_i(\alpha)} \) (\( W \) acting on \( \mathfrak{h}^* \)) \( \quad \)...so \( \text{Ad}(m_i) \) is of 1st kind.
- \( \{\text{Ad}(m_i)\}_{i \in I} \) satisfy same braid relations as \( \{r_i\}_{i \in I} \).
- \( \text{Ad}(m_i^2)|_{g_\alpha} = (-1)\alpha(h_i) \).

Given reduced decomposition \( w_\chi = r_{i_1} \cdots r_{i_\ell} \) we can define

\[
\text{Ad}(m_\chi) = \text{Ad}(m_{i_1}) \circ \cdots \circ \text{Ad}(m_{i_\ell}) \in \text{Aut}(g).
\]

- \( \text{Ad}(m_\chi)|_{\mathfrak{h}} = w_\chi. \)
- \( \text{Ad}(m_\chi)(g_\alpha) = g_{w_\chi(\alpha)}. \)
- \( \text{Ad}(m_\chi)|_{g_\chi} = \text{op}_X \circ \omega|_{g_\chi}. \)
- \( \text{Ad}(m_\chi^2)|_{g_\alpha} = (-1)^{2\alpha(\rho_\chi)} \) where \( \rho_\chi^\vee = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha^\vee \).

[Kac & Wang, 1992]
[Back-Valente, Bardy-Panse, Ben Massaoud & Rousseau, 1995]
Building special pseudo-involutions

Let \((X, \tau) \in \text{CDec}(A)\) and define

\[ \theta = \theta(X, \tau) := \text{Ad}(m_X) \circ \tau \circ \omega. \]

**Properties:**

- \(\theta|_{\mathfrak{h}} = -w_X \circ \tau.\)
- \(\theta(g_\alpha) = g_{\theta(\alpha)}\) with dual map \(\theta = -w_X \circ \tau \in \text{GL}(\mathfrak{h}^*).\)
- \(\theta|_{\mathfrak{g}_X} = \text{id}.\)
- \(\theta^2|_{\mathfrak{g}_\alpha} = (-1)^{2\alpha(\rho_X^\vee)}.\)

In particular, \(\theta\) is a pseudo-involution. Furthermore, every pseudo-involution is conjugate to \(\theta(X, \tau)\) for some \((X, \tau) \in \text{CDec}(A)\).
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The automorphism $\theta$ and the subalgebra $\mathfrak{k}$

Lemma (Kolb, 2014)

If $\theta \in \text{Aut}(g)$ is involutive and of 2nd kind then $\exists (X, \tau) \in \text{Sat}(A)$ s.t.

$$\theta = \text{scalar} \circ \text{Ad}(m_X) \circ \tau \circ \omega, \quad g^\theta = \langle g_X, h^\theta, \{f_i + \theta(f_i)\}_{i \in I \setminus X} \rangle.$$
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**Lemma (Kolb, 2014)**

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$$

**An optimist’s definition (Regelskis & Vlaar)**

For $(X, \tau) \in \text{CDec}(A)$ define $\theta = \theta(X, \tau) = \text{Ad}(m_X) \circ \tau \circ \omega$ and

$$
\mathfrak{k} = \mathfrak{k}(X, \tau) = \langle g_X, h^{\theta}, \{b_i\}_{i \in I \setminus X} \rangle = \langle n^{\perp}_{X}, h^{\theta}, \{b_i\}_{i \in I} \rangle
$$

where $n^{\perp}_{X} := \langle \{e_i\}_{i \in X} \rangle$ and $b_i := \begin{cases} f_i + \theta(f_i) & \text{if } i \in I \setminus X, \\ f_i & \text{if } i \in X. \end{cases}$
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$$\mathfrak{k} = \mathfrak{k}(X, \tau) = \langle g_X, h^\theta, \{b_i\}_{i \in I \setminus X} \rangle = \langle n^+_X, h^\theta, \{b_i\}_{i \in I} \rangle$$

where $n^+_X := \langle \{e_i\}_{i \in X} \rangle$ and $b_i := \begin{cases} f_i + \theta(f_i) & \text{if } i \in I \setminus X, \\ f_i & \text{if } i \in X. \end{cases}$

$$[h, h'] = 0, \quad [h, e_k] = \alpha_k(h)e_k, \quad [h, b_i] = -\alpha_i(h)b_i,$$

$$[e_k, b_i] = \delta_{ki} h_i, \quad \text{ad}(e_k)^{1-a_{kl}}(e_l) = 0 \text{ if } k \neq l$$

for $h, h' \in h^\theta, i, j \in I, k, l \in X$. What about a relation for $b_i, b_j \ (i \neq j)$?
Lemma (Regelskis & Vlaar)

Let \((X, \tau) \in \text{CDec}(A)\). For \(i, j \in I\) \((i \neq j)\) we have \(\text{ad}(b_i)^{1-a_{ij}}(b_j) = \begin{cases} 
2[\theta(f_i), [f_i, f_j]] & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j \in \Phi^-, a_{ij} = -1, \alpha_i(\rho^\vee_X) \in \mathbb{Z}, \\
-18e_j & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, a_{ij} = -3, \\
-(2h_i + h_j) & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, a_{ij} = -1, \\
2[\theta(f_i), f_j] & \text{if } \theta(\alpha_i) + \alpha_j \in \Phi^-, a_{ij} = 0, \alpha_i(\rho^\vee_X) \in \mathbb{Z}, \\
h_i - h_j & \text{if } \theta(\alpha_i) + \alpha_j = 0, a_{ij} = 0, \\
4b_i & \text{if } \theta(\alpha_i) + \alpha_j = 0, a_{ij} = -1, \\
\sum_{r=1}^\lfloor(1-a_{ij})/2\rfloor p_{ij}^{(r)} \text{ad}(b_i)^{1-a_{ij}-2r}(b_j) & \text{with } p_{ij}^{(r)} \in \mathbb{Z} \text{ recursively defined,} \\
0 & \text{otherwise.} 
\end{cases}\)

Proof.

Recursive root space analysis of \(\text{ad}(b_i)^m(b_j)\) for \(m \in \mathbb{Z}_{\geq 0}\); some computations; set \(m = 1 - a_{ij}\) and use \(\text{ad}(f_i)^{1-a_{ij}}(f_j) = 0\).
Lemma (Regelskis & Vlaar)

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2[\theta(f_i), [f_i, f_j]] \in n^+_X & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j \in \Phi^-, a_{ij} = -1, \alpha_i(\rho^\vee_X) \in \mathbb{Z}, \\
-18e_j \in n^+_X & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, a_{ij} = -3, \\
-(2h_i + h_j) \notin \mathfrak{h}^\theta & \text{if } \theta(\alpha_i) + \alpha_i + \alpha_j = 0, a_{ij} = -1, \\
2[\theta(f_i), f_j] \in n^+_X & \text{if } \theta(\alpha_i) + \alpha_j \in \Phi^-, a_{ij} = 0, \alpha_i(\rho^\vee_X) \in \mathbb{Z}, \\
h_i - h_j \in \mathfrak{h}^\theta & \text{if } \theta(\alpha_i) + \alpha_j = 0, a_{ij} = 0, \\
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\sum_{r=1}^{\lfloor(1-a_{ij})/2\rfloor} p_{ij}^{(r)} \text{ad}(b_i)^{1-a_{ij}-2r} b_j & \text{with } p_{ij}^{(r)} \in \mathbb{Z} \text{ recursively defined,} \\
0 & \text{if } \theta(\alpha_i) + \alpha_i = 0, j \in I \setminus X, \\
otherwise.
\end{cases}
\]

Proof.

Recursive root space analysis of \(\text{ad}(b_i)^m(b_j)\) for \(m \in \mathbb{Z}_{\geq 0}\); some computations; set \(m = 1 - a_{ij}\) and use \(\text{ad}(f_i)^{1-a_{ij}}(f_j) = 0\).
Generalized Satake diagrams, pseudo-fixed-pt. subalgebras

\[ \text{GSat}(A) := \{(X, \tau) \in \text{CDec}(A) | \forall i \in I \setminus X \cup \{i, \tau(i)\} \text{ does not have } o \in X \text{ as a connected component}\}. \]

Satake diagrams are examples of generalized Satake diagrams.
Generalized Satake diagrams, pseudo-fixed-pt. subalgebras

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Satake diagrams are examples of generalized Satake diagrams.
Using the \( \text{ad}(b_i)^{1-a_{ij}}(b_j) \) lemma we can now prove:

**Theorem (Regelskis & Vlaar)**

*For \((X, \tau) \in CDec(A), the following are equivalent:*

1. \((X, \tau) \in \text{GSat}(A)\).
2. \(\text{ad}(b_i)^{1-a_{ij}}(b_j) \in n_X^+ \oplus h^\theta \oplus (\text{lower order terms in } b_k)\).
3. \(\mathfrak{k} \cap h = h^\theta\),
   *for all \(\alpha \in \Phi\): \(g_\alpha \subset \mathfrak{k}\) if and only if \(\theta(\alpha) = \alpha\),
   *and for all \(\alpha \in \Phi\): \(g_\alpha + g_{\theta(\alpha)}\) intersects \(\mathfrak{k}\) nontrivially.*
Let $A$ be of finite type.

- If $A$ is of type $A_n$ then $\text{GSat}(A) = \text{Sat}(A)$.
- $\text{GSat}(A) \setminus \text{Sat}(A)$ consists of the following:

  Here $i$ is the unique node such that $i = \tau(i)$ and $\alpha_i(\rho_X^\vee) \notin \mathbb{Z}$.

For $A$ of affine type, $\text{GSat}(A) = \text{Sat}(A)$ if and only if $A$ is of type $\widehat{A}_n$. 
Structure of $\mathfrak{k}$ if $\text{dim}(g) < \infty$

Fixed-point subalgebras of semisimple automorphisms are reductive, e.g. [Jacobson, 1962]. For the “new” subalgebras, using the $\text{ad}(b_i)^{1-a_{ij}}(b_j)$ lemma one obtains

**Theorem (Regelskis & Vlaar)**

Let $A$ be of finite type. If $(X, \tau) \in \text{GSat}(A) \setminus (\text{Sat}(A) \cup \{\circlearrowright\})$ then

$$\mathfrak{k} = (\text{nilpotent of class } 2) \rtimes (\text{reductive with abelian part } \leq 1\text{-dim'})$$
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If $(X, \tau) = \bullet \rightarrow \circ$ then $\mathfrak{k} \cong (3\text{-dim. Heisenberg Lie algebra}) \rtimes \mathfrak{sl}_2$. 
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If $(X, \tau) = \circ \longrightarrow \bullet$ then (with some work) $\mathfrak{k} \cong \mathfrak{sl}_3$ and (with some more work) one checks also here $\mathfrak{k} \neq g^\sigma$. 
Involutive automorphisms of root systems

For $A$ of finite type, generalized Satake diagrams appeared already in [A. Heck, 1984]. For $(X, \tau) \in \text{CDec}(A)$ recall $\theta = -w_X \tau \in \text{GL}(\mathfrak{h}^*)$ and consider $W^\theta := \{ w \in W | w\theta = \theta w \}$. Let $V = \mathbb{R}\Phi$, decompose $V = V^\theta \oplus V^{-\theta}$ and denote the projection onto $V^{-\theta}$ by $\bar{\cdot}$. 
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Restricted root system and the restricted Weyl group:

$$\Phi := \{ \alpha \mid \alpha \in \Phi \} \setminus \{0\},$$

$$\overline{W} := \{ w_{|_{V^{-\theta}}} \mid w \in W \text{ s.t. } w(V^{-\theta}) \subseteq V^{-\theta} \} \cong W^\theta / W_X.$$

**Theorem (A. Heck)**

$$(X, \tau) \in \text{GSat}(A) \iff \forall i \in I \setminus X : \text{op}_{X \cup \{i, \tau(i)\}}(X) = X \iff \overline{W} = W(\Phi)$$

Moreover, by [Lusztig, 1976] this is equivalent to $\{ w_X w_{X \cup \{i, \tau(i)\}} \}_{i \in I \setminus X}$ being a Coxeter system for the subgroup $\widetilde{W} \subset W^\theta$ it generates.
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- The quasi K-matrix factorizes like the longest element of $\widetilde{W}$ [Dobson & Kolb, 2018].
1 Introduction and review

2 Pseudo-involutions and compatible decorations

3 Pseudo-fixed-point subalgebras and generalized Satake diagrams

4 Quantized pseudo-fixed-point subalgebras
Consider $U_q\mathfrak{g} = \langle E_i, F_i, k_i^{\pm 1}, \ldots \rangle$. For $(X, \tau) \in \text{CDec}(A)$ recall

$\theta = \theta(X, \tau) = \text{Ad}(m_X) \tau \omega$. For $\gamma, s \in \mathbb{C}^{I \setminus X}$ we define

$$B_{\gamma, s}(X, \tau) = \langle U_q\mathfrak{g}X, U_q(\mathfrak{h}^\theta), \{B_i\}_{i \in I \setminus X} \rangle \subseteq U_q\mathfrak{g}$$

where

$$B_i = F_i + \gamma_i \text{Ad}(T_X)(E_{\tau(i)}) k_i^{-1} + s_i k_i^{-1}.$$ 

- $\gamma_i = c_i \times$ (scalar needed in definition of involutive $\theta$)
- $B_{\gamma, s}(X, \tau)$ is a right coideal subalgebra which tends to $U\mathfrak{k}(X, \tau)$ as $q \to 1$.
- If $(X, \tau) \in \text{GSat}(A)$ then, for suitable $\gamma, s$,

$$B_{\gamma, s}(X, \tau) \cap U_q\mathfrak{h} = U_q(\mathfrak{h}^\theta)$$
Universal K-matrix and a conjecture

Take $A$ of finite type and $(X, \tau) \in \text{GSat}(A)$. Then $B = B_{\gamma,s}(X, \tau)$ is quasitriangular, i.e. $\exists$ invertible $K \in \hat{U}_q\mathfrak{g}$ (completion w.r.t. $\mathcal{O}_{\text{int}}$)

$$Kb = bK \quad \text{for all } b \in B$$

$$\mathcal{R}(1 \otimes K)\mathcal{R} \in \text{completion of } B \otimes U_q\mathfrak{g}$$

$$\Delta(K) = (K \otimes 1)\mathcal{R}_{21}(1 \otimes K)\mathcal{R}.$$
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Conjecture

All quasitriangular coideal subalgebras of $U_q\mathfrak{g}$ are of the form $B_{\gamma,s}(X, \tau)$ with $(X, \tau) \in \text{GSat}(A)$. 

Evidence \cite{Mudrov, 2002} for $U_q\mathfrak{sl}_N$ classified all solutions to the matrix reflection equation (vector rep.). Each of them intertwines a unique $B_{\gamma,s}(X, \tau)$.

The same appears to happen for other types (verified in low rank).
Universal $K$-matrix and a conjecture

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All quasitriangular coideal subalgebras of $U_q g$ are of the form $B_{\gamma,s}(X, \tau)$ with $(X, \tau) \in \text{GSat}(A)$.

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